2023.M29

2023L003A1EL

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination 2023

Mathematics

Paper 1

Higher Level

Friday 9 June   Afternoon 2:00 - 4:30

300 marks

Examination Number

Day and Month of Birth

For example, 3rd February is entered as 0302

Centre Stamp

The 2023 examination papers were adjusted to compensate for disruptions to learning due to COVID-19. This examination paper does not necessarily reflect the same structure and format as the examination papers of past or subsequent years.

Do not write on this page

Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	4 questions

Answer questions as follows:

- **any five** questions from Section A – Concepts and Skills
- **any three** questions from Section B – Contexts and Applications.

Write your Examination Number in the box on the front cover.

Write your answers in blue or black pen. You may use pencil in graphs and diagrams only.

This examination booklet will be scanned and your work will be presented to an examiner on screen. Anything that you write outside of the answer areas may not be seen by the examiner.

Write all answers into this booklet. There is space for extra work at the back of the booklet. If you need to use it, label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

In general, diagrams are not to scale.

You will lose marks if your solutions do not include relevant supporting work.

You may lose marks if the appropriate units of measurement are not included, where relevant.

You may lose marks if your answers are not given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	150 marks

Answer **any five** questions from this section.

Question 1 **(30 marks)**

(a) Find the two values of $m \in \mathbb{R}$ for which $|5 + 3m| = 11$.

(b) For the real numbers h, j, and k:

$$\frac{1}{h} = \frac{k}{j + k}$$

Express k in terms of h and j.

(c) $x^2 - px + 1$ is a factor of $x^3 - 2x - 3r$, where $p, r \in \mathbb{R}$ and $p < 0$.

Find the value of p and the value of r.

$p =$ ____________ and $r =$ ____________

Question 2 **(30 marks)**

(a) $f(x) = x^2 + bx + c$, where $b, c \in \mathbb{R}$.
$f(x)$ has a local minimum point at $(3, -1)$.

Find the value of b and the value of c.

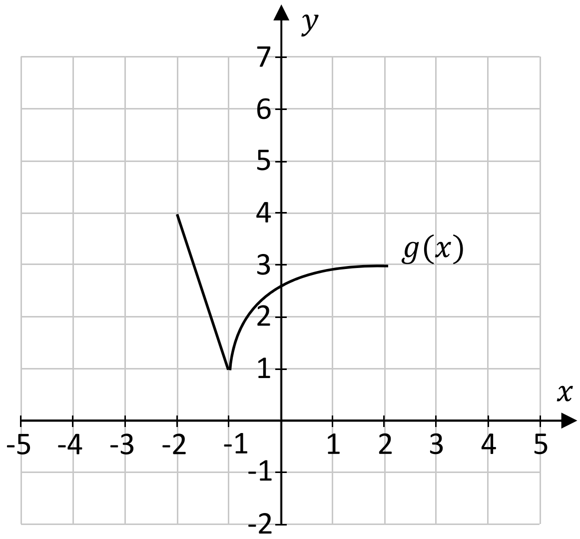
$b =$ ____________ and $c =$ ____________

(b) Find the value of the following limit, where $n \in \mathbb{N}$:

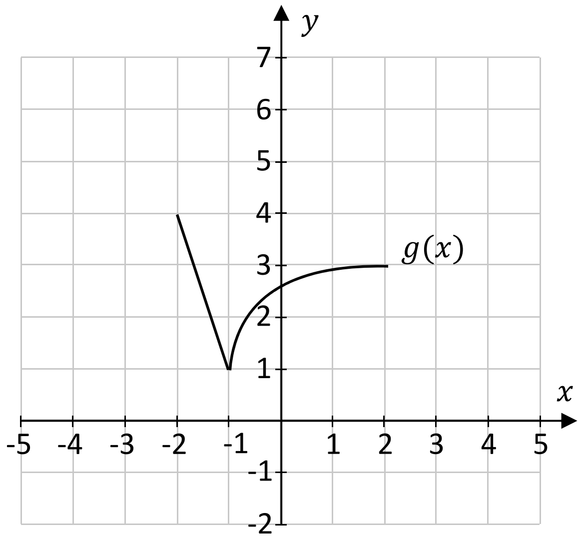
$$\lim_{n \to \infty} \left[\frac{n}{n+1} + \frac{n+1000}{n} + \left(\frac{1}{3}\right)^n \right]$$

(c) The function $g(x)$ is defined for $-2 \leq x \leq 2$, $x \in \mathbb{R}$.
Its graph is shown in each of the two diagrams below.

(i) Draw the graph of $g(x) - 2$ on the co-ordinate diagram below, for $x \in \mathbb{R}$, on as large a domain as possible.



(ii) Draw the graph of $g(x + 3)$ on the co-ordinate diagram below, for $x \in \mathbb{R}$, on as large a domain as possible.



Question 3 **(30 marks)**

(a) Prove that $\sqrt{2}$ is **not** a rational number.

(b) t is a positive real number, with:

$$\log_3 t + \log_9 t + \log_{27} t + \log_{81} t = 10$$

Find the value of t. Give your answer in the form 3^r, where $r \in \mathbb{Q}$.

Hint: use the formula $\log_a b = \dfrac{\log_c b}{\log_c a}$.

(c) **(i)** Explain what $\log_6 m$ means, where m is a positive real number.

(ii) m is a real number, and $m > 6$.

What information does this give about the value of $\log_6 m$?

Question 4 **(30 marks)**

In this question, $i^2 = -1$.

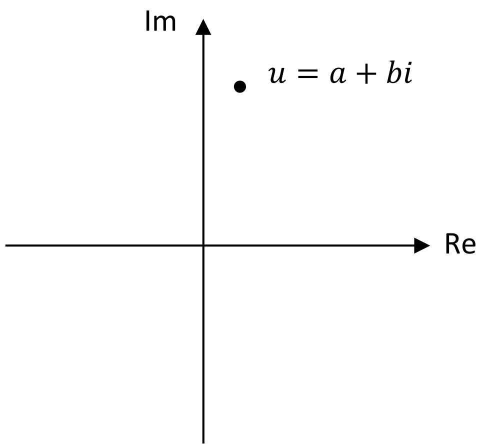
(a) The complex number $z_1 = 1 + i$ is a root of the equation $z^2 + (3 - 2i)z + p = 0$.

Find the value of p, where $p = a + bi$, with $a, b \in \mathbb{Z}$.

(b) Use **De Moivre's Theorem** to find the values of w for which $w^2 = -1 + \sqrt{3}i$.

Give each value of w in the form $a + bi$, with $a, b \in \mathbb{R}$.

(c) The Argand diagram below shows the complex number $u = a + bi$, where $a, b \in \mathbb{R}$.



(i) Write the complex numbers iu and $\overline{iu}$ in their simplest form, in terms of a and b, where $\overline{iu}$ is the complex conjugate of iu.

iu:

$\overline{iu}$:

(ii) **Plot** and **label** the complex numbers iu **and** $\overline{iu}$ on the diagram above, as accurately as possible.

(iii) State a transformation, or series of transformations, that would send u to $\overline{iu}$.
Do **not** include a translation in your answer.

Question 5 **(30 marks)**

(a) The function f is defined as follows, for $x \in \mathbb{R}$:

$$f(x) = \frac{1}{5x^2 + 7}$$

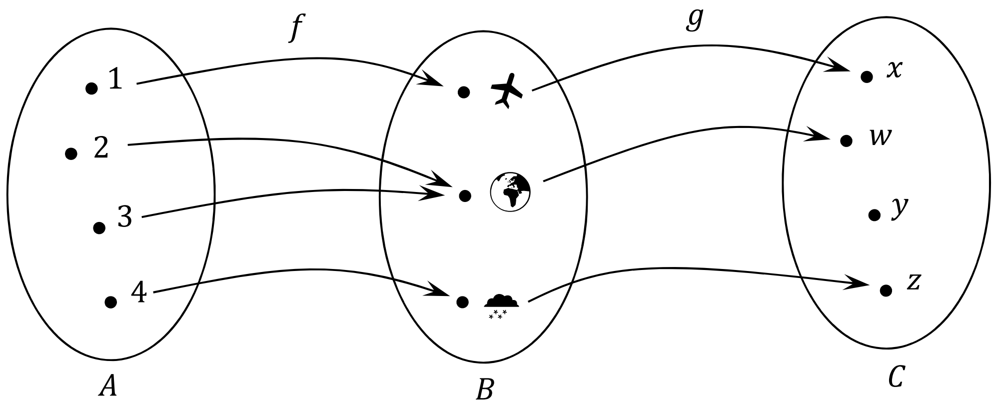
Find $f'(x)$, the derivative of f. Give your answer in its simplest form.

(b) The function $g(x)$ is defined as follows, for $x \in \mathbb{R}$, $0 < x < \pi$:

$$g(x) = \left(\tan\left(\frac{x}{2}\right)\right)(\ln x)$$

Find the value of $g'\left(\frac{\pi}{2}\right)$. Give your answer in the form $a + \ln b$, where $a, b \in \mathbb{R}$.

(c) The diagram below shows three sets, A, B, and C, and two functions, f and g, where $f: A \to B$ and $g: B \to C$. $\#A = \#C = 4$ and $\#B = 3$.



(i) Find the value of $g\big(f(3)\big)$.

(ii) Explain why $g: B \to C$ is injective but **not** surjective (that is, one-to-one but **not** onto).

Injective:

Not surjective:

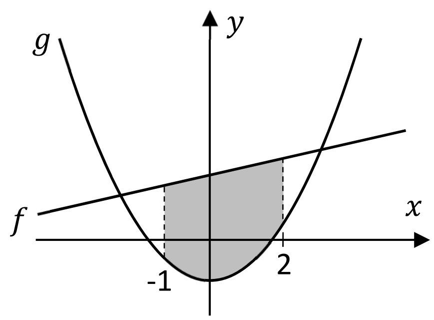
Question 6 **(30 marks)**

(a) f and g are two functions of $x \in \mathbb{R}$, where:

$$f(x) = x + 4$$
$$g(x) = x^2 - 2$$

(i) Find the two values of x for which $f(x) = g(x)$.

(ii) Find the **area** of the shaded region in the diagram below (not to scale), the region between the graphs of $f(x) = x + 4$ and $g(x) = x^2 - 2$, from $\boldsymbol{x} = \mathbf{-1}$ to $\boldsymbol{x} = \mathbf{2}$.



(b) $b \in \mathbb{R}$ is a positive constant, and:

$$\int_0^b b\, e^{bx}\, dx = e$$

Work out the value of b.

Section B — Contexts and Applications — 150 marks

Answer **any three** questions from this section.

Question 7 **(50 marks)**

Fiona is driving on a motorway. She passes a point **A** on the motorway. Her speed is given by:

$$v(t) = \frac{2}{3}t^3 - 6t^2 + 13t + 109$$

where v is her speed in km/hour t minutes after passing the point **A**, for $0 \leq t \leq 5$ and $t \in \mathbb{R}$.

(a) Work out Fiona's speed when she passes the point **A**.

(b) Work out Fiona's acceleration (that is, the rate at which her speed is increasing) 5 minutes after she passes the point **A**. Give your answer in km/hour per minute.

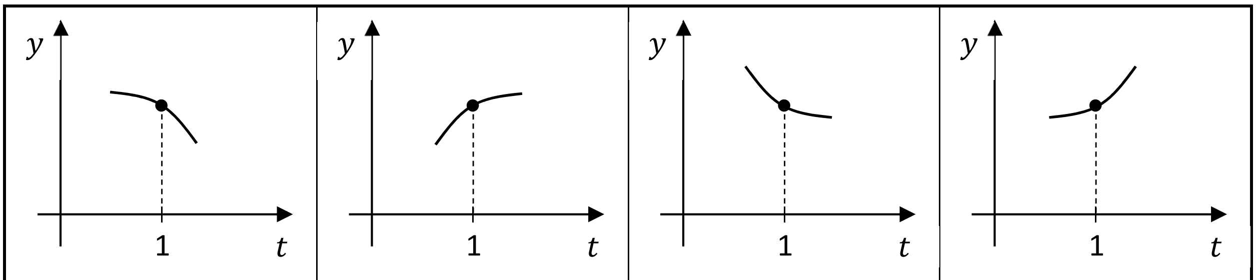
(c) Find the time (value of t) at which Fiona reaches her maximum speed, during the first 4 minutes after she passes the point **A**. Give your answer correct to 2 decimal places.

(d) Use integration to work out Fiona's average speed over the 5 minutes after she passes the point **A**. Give your answer correct to 2 decimal places.

(e) Taking $v'(t)$ to be the derivative of v, and $v''(t)$ to be the second derivative of v:

$$v'(1) > 0 \text{ and } v''(1) < 0$$

Four graphs, **A**, **B**, **C**, and **D**, are shown below.



Graph A **Graph B** **Graph C** **Graph D**

Close to where $t = 1$, the graph of $y = v(t)$ must look like one of the four graphs given above. Write down which graph this is. Justify your answer, using both $v'(1)$ **and** $v''(1)$.

Answer (**A**, **B**, **C**, or **D**):

Using $v'(1) > 0$:

Using $v''(1) < 0$:

This question continues on the next page.

There is an **Average Speed Zone** on the motorway, starting at the point **A** and ending at point **B**.
The distance from **A** to **B** along the motorway is 10 km.
Cameras record the time taken for each car to travel from the point **A** to the point **B**.
Each car's average speed from **A** to **B** is then calculated.

(f) Work out the **minimum** time, in minutes, that a driver could get from **A** to **B**, while not driving above 100 km/hour.

(g) Rohan drives from **A** to **B**.

He passes the point **A** driving at a constant speed of 120 km/hour. After 2 minutes driving at this speed, he starts to decelerate (reduce his speed) at a constant rate, until he reaches the point **B**. Overall, his average speed in driving from **A** to **B** is 100 km/hour.

Work out Rohan's deceleration. Give your answer in km/hour per minute.

Question 8 **(50 marks)**

Olga, Chen, Fiona, and Rohan all have bank accounts.

(a) Olga puts €3000 in a savings account. Interest is added annually at a rate of $2{\cdot}4\%$ per year. Work out the amount in Olga's account after 5 years, correct to the nearest cent.

(b) **(i)** Explain what is meant by the "present value" of a payment of €1000 in 1 year's time, at a particular interest rate.

(ii) Chen puts a different amount in a savings account with the same interest rate ($2{\cdot}4\%$ per year). After 6 years, Chen has €4000 in the account.

Work out how much money Chen put in the account initially, correct to the nearest cent.

This question continues on the next page

(c) Fiona is taking out a loan at the same annual interest rate ($2{\cdot}4\%$ per year).
Fiona makes payments quarterly (that is, 4 times per year).
Work out the quarterly interest rate that would be equivalent to an APR of $2{\cdot}4\%$.
Give your answer as a **percentage**, correct to 2 decimal places.

(d) Rohan wants to put the same amount of money in a savings account at the start of each month for 36 months so that, at the end of 3 years, he will have a total of €$12\ 000$ in the account. Interest is calculated at a rate of $0{\cdot}11\%$ per month.

(i) Taking €A to be the amount Rohan puts in his account at the start of each month, write down a geometric series in €A to show the total amount of money in the account at the end of the 3 years. Include the first two and the last two terms.

(ii) Hence, find the value of €A that will give a total of €$12\ 000$ in the account after 3 years. Give your answer correct to the nearest cent.

(e) A park sells three types of ticket: child, student, and adult. The table below gives information on the price of each ticket and the percentage of tickets sold.
For example, 15% of all tickets sold are student tickets.

Type of ticket	**Child**	**Student**	**Adult**
Price of ticket	€11	€5 less than an adult ticket	€ x
Percentage	52%	15%	33%

The expected value of the price of a ticket is €$13{\cdot}85$.
Work out the value of x, the price of an adult ticket.

(f) When an item is being sold:

- the **mark up** is the profit as a percentage of the cost price, and
- the **margin** is the profit as a percentage of the selling price.

A shop sells an item with a **margin** of 18%. Work out the **mark up** for this item.
Give your answer as a percentage, correct to the nearest percent.

Question 9 **(50 marks)**

Ava is investigating factors of different numbers.

(a) First, she looks at numbers that can be written as powers of a prime number.

(i) List the 5 different factors of 2^4. You can write each one as a power of 2.

(ii) Work out how many different factors 3^7 has.

(iii) Work out how many different factors $2^{10} \times 3^{12}$ has.

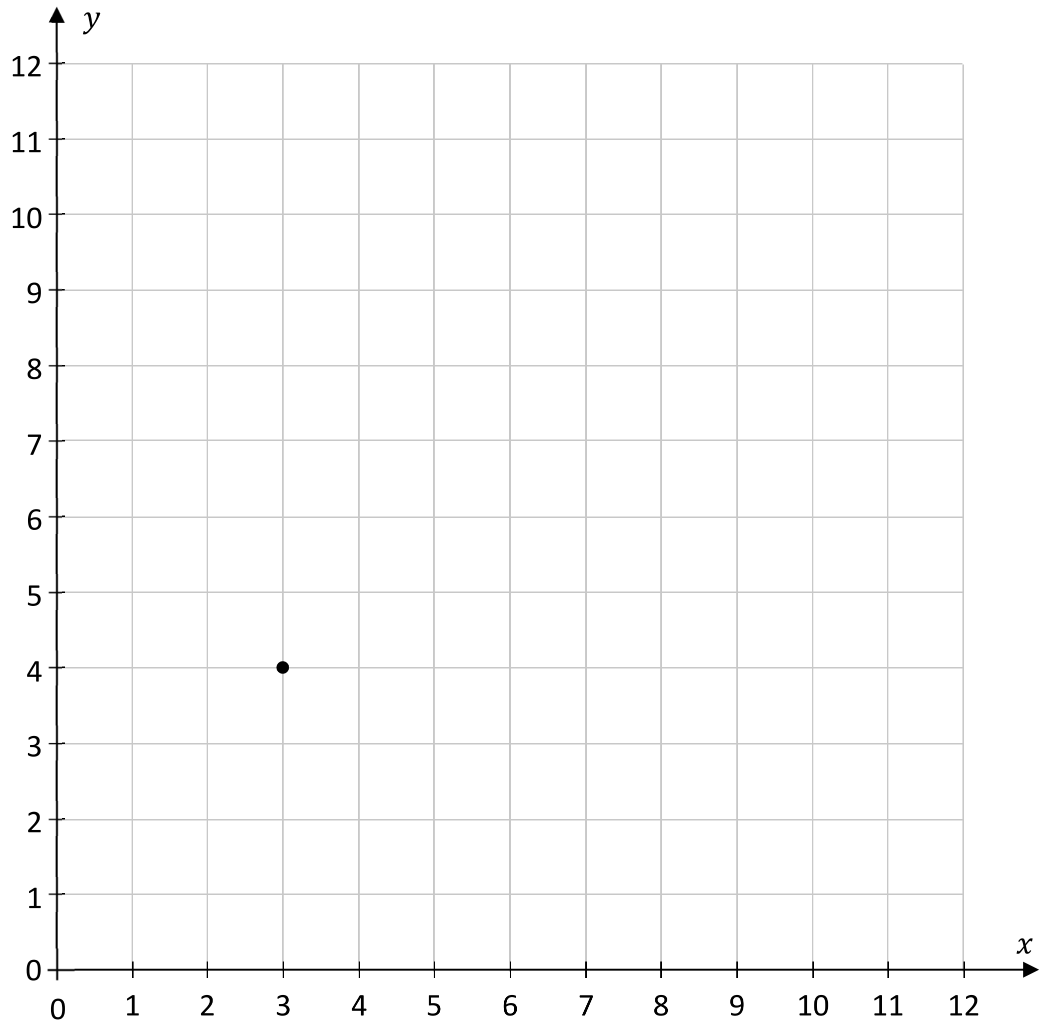
Ava also looks at the relationship between pairs of factors of a number.

(b) She makes a table to show the pairs of factors of 12 (that is, the pairs of natural numbers x and y with $xy = 12$).

(i) Complete the table below, showing the 6 pairs of factors (x and y) of 12.

x	1	2	3		6	
y			4			1

(ii) Plot the 6 points above on the co-ordinate diagram below. One of the points is shown.



(iii) Ava realises that the relationship between x and y is $y = \dfrac{12}{x}$.

On the co-ordinate diagram above, **draw** the graph of $y = \dfrac{12}{x}$ for $x \in \mathbb{R}$, in the domain $1 \le x \le 12$.

This question continues on the next page.

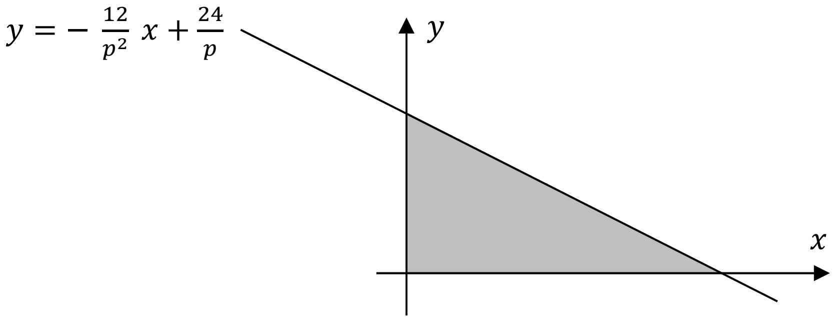
(c) **(i)** A tangent to the curve $y = \frac{12}{x}$ is drawn at the point $\left(p,\ \frac{12}{p}\right)$, where $p \in \mathbb{R}$ and $p > 0$.

Show that the equation of this tangent is:

$$y = -\frac{12}{p^2}x + \frac{24}{p}$$

(ii) The **area of the triangle** formed by the x-axis, the y-axis, and the tangent $y = -\frac{12}{p^2}x + \frac{24}{p}$ is always k square units, where $k \in \mathbb{N}$ is a constant.

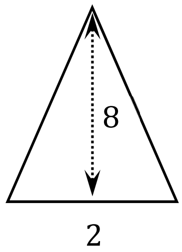
Work out the value of k.



$k =$ ____________

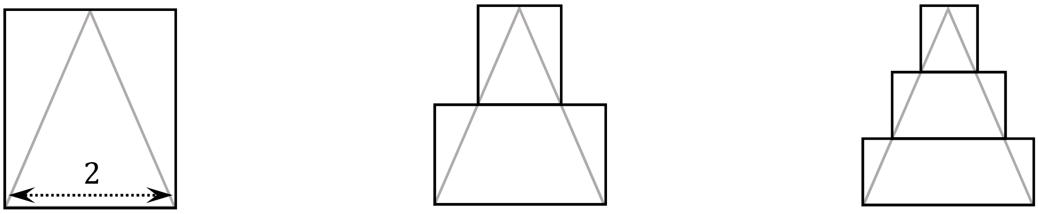
Question 10 **(50 marks)**

A triangle has a base of length 2 units and a perpendicular height of 8 units, as shown in the diagram on the right.



The diagrams below show T_1, T_2, and T_3, the first three shapes in a sequence of shapes based on this triangle.

For each value of $n \in \mathbb{N}$, the shape T_n is made up of n rectangles of equal height laid on top of each other. T_n is the collection of the smallest such rectangles that completely covers the triangle.

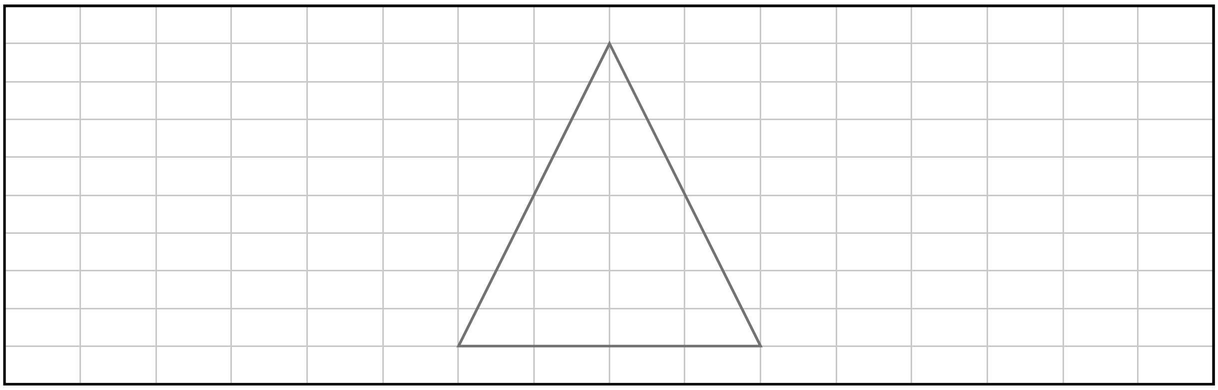


T_1
1 rectangle

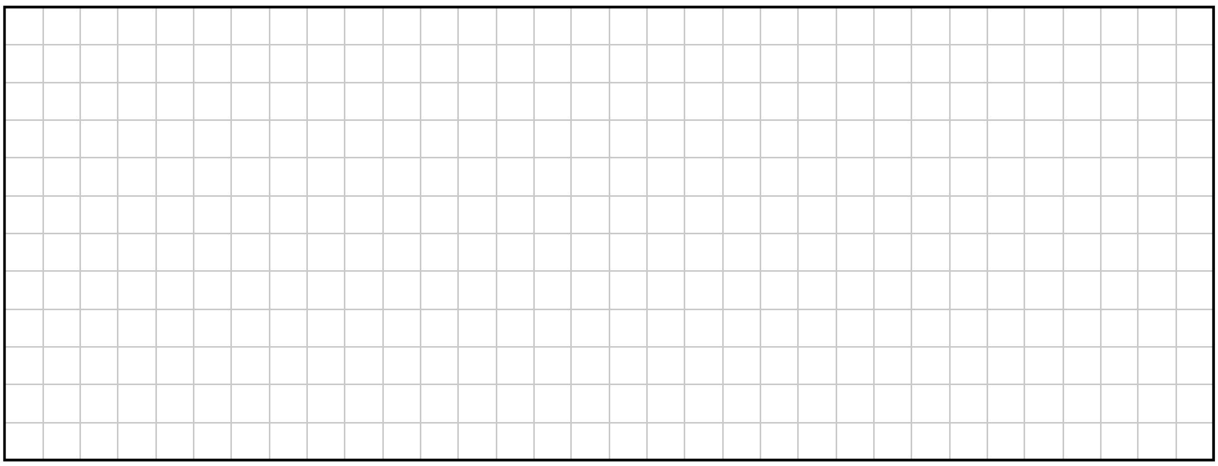
T_2
2 rectangles
of equal height

T_3
3 rectangles
of equal height

(a) Draw T_4 in the grid below, based on the triangle given on the grid.



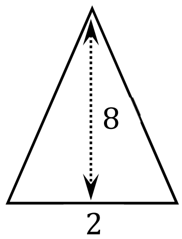
(b) Show that the **total area** of the three rectangles in T_3 is $\frac{32}{3}$ square units.



(c) Find the **total area** of the n rectangles in T_n , for $n \in \mathbb{N}$.
Give your answer in square units in terms of n, in its simplest form.

(d) The total area of the rectangles in the nth term of a **different** sequence of groups of rectangles is as follows, for $n \in \mathbb{N}$:

$$\text{Total area} = A_n = \frac{8\,(n-1)}{n}$$



Work out the first value of n for which A_n is **greater than** 95% of the area of the triangle on the right.

This question continues on the next page.

(e) **Diagram A** below shows a square-based pyramid, with base sides of length c units.
The base of the pyramid is horizontal, and its perpendicular height is h units (where $c, h \in \mathbb{R}$).

Diagram B below shows the same pyramid. It also shows a horizontal square that lies within the pyramid, a distance of x units down from the top of the pyramid, where $x \in \mathbb{R}$, $0 < x < h$.

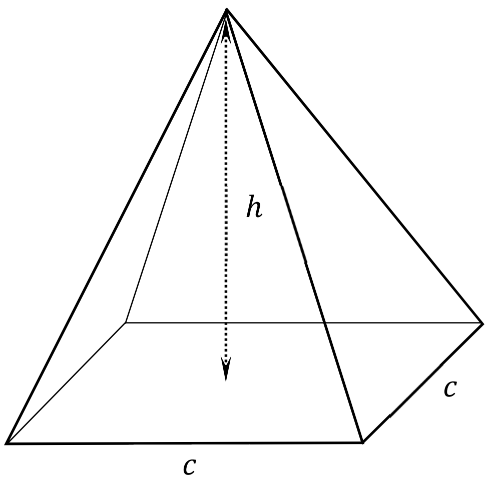


Diagram A

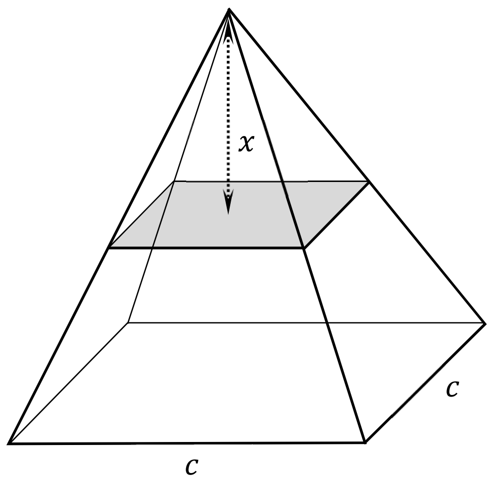


Diagram B

The area of the shaded square in **Diagram B** is $S(x) = \dfrac{x^2\,c^2}{h^2}$.

(i) The volume of the pyramid is $\int_0^h S(x)\,dx$.

Use integration to find the volume of the pyramid in cubic units, in terms of c and h.

(ii) x starts to increase at a rate of 3 units per second.
This causes $S(x)$ to increase as well.

Find the rate of change of $S(x)$ with respect to time, at the instant when x is half the perpendicular height of the pyramid. Give your answer in square units per second, in terms of c and h.

Page for extra work.
Label any extra work clearly with the question number and part.

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Leaving Certificate – Higher Level

Mathematics Paper 1

Friday 9 June

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