2024 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	C
3	A
4	B
5	D
6	A
7	C
8	A
9	B
10	D

Section II

Question 11 (a)

Criteria	Marks
• Provides correct primitive	2
• Attempts to use integration by parts	1

Sample answer:

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad u = x \quad v' = e^x$$

$$= xe^x - e^x + C \qquad u' = 1 \quad v = e^x$$

Question 11 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$z = 2 + 3i, \quad w = 1 - 5i$$

$$z + \bar{w} = 2 + 3i + 1 + 5i$$

$$= 3 + 8i$$

Question 11 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$z^2 = (2 + 3i)^2$$

$$= 4 + 12i + 9i^2$$

$$= -5 + 12i$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	2
• Finds $\underset{\sim}{u} \cdot \underset{\sim}{v}$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta$$

$$4 - 8 - 14 = \sqrt{1+4+4} \cdot \sqrt{16+16+49}\cos\theta$$

$$-18 = 3 \cdot 9\cos\theta$$

$$\cos\theta = -\frac{2}{3}$$

$$\theta = 2.3$$

Question 11 (d)

Criteria	Marks
• Provides correct solution	3
• Obtains correct integral, or equivalent merit	2
• Attempts to use the t substitution, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta + 1}\,d\theta = \int_0^1 \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{1+t^2}{1+t^2}}$$

$$= 2\int_0^1 \frac{1}{(t+1)^2}\,dt$$

$$= 2\int_0^1 (t+1)^{-2}\,dt$$

$$= 2\left[\frac{(t+1)^{-1}}{-1}\right]_0^1$$

$$= -2\left[\frac{1}{t+1}\right]_0^1$$

$$= -2\left(\frac{1}{2}\right) + 2\left(\frac{1}{1}\right)$$

$$= -1 + 2$$

$$= 1$$

$$t = \tan\frac{\theta}{2}$$

$$\theta = 0 \quad t = 0$$

$$\theta = \frac{\pi}{2} \quad t = \tan\frac{\pi}{4} = 1$$

Question 11 (e) (i)

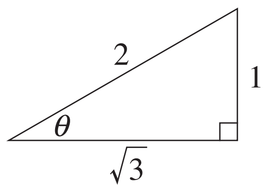
Criteria	Marks
• Provides correct solution	2
• Obtains correct modulus or argument, or equivalent merit	1

Sample answer:

$$\sqrt{3} + i = 2(\cos\theta + i\sin\theta)$$

$$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 2e^{i\frac{\pi}{6}}$$



$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

Question 11 (e) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains correct number in modulus-argument form, or equivalent merit	1

Sample answer:

$$\left(\sqrt{3} + i\right)^7 = \left(2e^{i\frac{\pi}{6}}\right)^7$$

$$= 128e^{i\frac{7\pi}{6}}$$

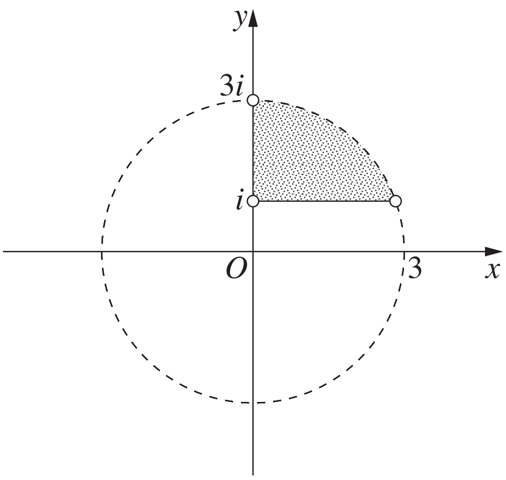
$$= \frac{128}{2}\left(-\sqrt{3} - i\right)$$

$$= -64\sqrt{3} - 64i$$

Question 11 (f)

Criteria	Marks
• Provides correct sketch	3
• Provides correct shading of the interior of the region, or equivalent merit	2
• Provides correct sketch of interior of either $\lvert z \rvert < 3$ or $0 \le \arg(z - i) \le \frac{\pi}{2}$, or equivalent merit	1

Sample answer:



Question 12 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \cdot \underset{\sim}{b} = -\frac{10}{20}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Question 12 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds $\underset{\sim}{a} - \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$, or equivalent merit	1

Sample answer:

$$\left(\underset{\sim}{a} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}\right) \cdot \underset{\sim}{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = 4 + 0 - 4 = 0$$

$\therefore$ Vectors are perpendicular.

Question 12 (b)

Criteria	Marks
• Provides correct primitive	3
• Obtains correct integrand, or equivalent merit	2
• Attempts partial fractions using $\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$, or equivalent merit	1

Sample answer:

$$\int \frac{3x^2+2x+1}{(x-1)(x^2+1)}\,dx$$

$$\frac{3x^2+2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$3x^2+2x+1 = A(x^2+1)+(Bx+C)(x-1)$$

$x=1$ $\quad 2A = 6$

$\therefore \quad A = 3$

$x=0$ $\quad A - C = 1$

$\therefore \quad C = 2$

$x=-1$ $\quad 2A + (-B+C)(-2) = 2$

$6 + 2B - 4 = 2$

$2B = 0$

$\therefore \quad B = 0$

$$\therefore \quad \int \frac{3x^2+2x+1}{(x-1)(x^2+1)}\,dx = \int \frac{3}{x-1} + \int \frac{2}{x^2+1}\,dx$$

$$= 3\ln|x-1| + 2\tan^{-1}x + C$$

Question 12 (c) (i)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

$|z| = z + 8 + 12i$

LHS real so $a + bi + 8 + 12i$ real

$\therefore \quad (b + 12)i = 0$

$b = -12$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains quadratic equation for a, or equivalent merit	1

Sample answer:

$$|a - 12i|^2 = (a + 8)^2$$

$$\cancel{a^2} + 144 = \cancel{a^2} + 16a + 64$$

$$80 = 16a$$

$$a = 5$$

$$z = 5 - 12i$$

Question 12 (d)

Criteria	Marks
• Provides correct explanation	2
• Recognises that considering n odd/even is a correct approach OR • Proves the result for n odd or even, or equivalent merit	1

Sample answer:

$(n+1)^{41} - 79n^{40} = 2$

If n is odd, then $(n+1)$ is even.

$\therefore \quad (n+1)^{41}$ is even and $79n^{40}$ is odd.

But, even $-$ odd $\neq$ even

If n is even, then $(n+1)$ is odd.

$\therefore \quad (n+1)^{41}$ is odd and $79n^{40}$ is even.

But, odd $-$ even $\neq$ even

$\therefore$ Original statement cannot be true.

Question 12 (e) (i)

Criteria	Marks
• Provides correct equation	1

Sample answer:

$A(3, 5, -4)$ and $B(7, 0, 2)$

$$\text{A direction vector } = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$$

$$\ell: \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$$

Question 12 (e) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains one correct λ value for one coordinate, or equivalent merit	1

Sample answer:

$$\begin{pmatrix} 10 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$$

$10 = 3 + 4\lambda \qquad \lambda = \dfrac{7}{4}$

$5 = 5 - 5\lambda \qquad \lambda = 0$

Inconsistent values for λ

$\therefore$ Point $(10, 5, -2)$ is not on ℓ.

Question 13 (a) (i)

Criteria	Marks
• Provides correct proof	1

Sample answer:

$$AB = \begin{pmatrix} p-8 \\ p+8 \\ 2p-5 \end{pmatrix}$$

$$\begin{aligned} |AB|^2 &= p^2 - 16p + 64 + p^2 + 12p + 36 + 4p^2 - 20p + 25 \\ &= 6p^2 - 24p + 125 \end{aligned}$$

Question 13 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds the correct value of p, or equivalent merit	1

Sample answer:

$|AB|^2$ is a quadratic equation.

Axis of symmetry $\quad x = \dfrac{-b}{2a}$

$$\therefore \quad p = \frac{-(-24)}{2 \times 6}$$

$$= 2$$

The parabola is concave up.

Shortest distance (minimum) when $p = 2$

$$\begin{aligned} \therefore \quad \text{dist}^2 &= 6(2)^2 - 24(2) + 125 \\ &= 24 - 48 + 125 \\ &= 101 \end{aligned}$$

$\therefore$ Shortest distance is $\sqrt{101}$. (distance will be positive)

Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain v^2, or equivalent merit	2
• Find correct period or n, or equivalent merit	1

Sample answer:

$$\ddot{x} = -4\big(x - (-1)\big)$$

$\therefore\ n^2 = 4$

$\therefore\ n = 2 \quad \text{and} \quad c = -1$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4(x+1)$$

Integrate from $x = 0,\ r = 4$ to $x = a - 1,\ v = 0$

$$v^2 = 4\left(A^2 - (x+1)^2\right)$$

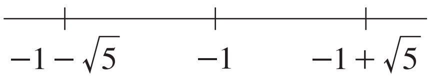
$x = 0, \quad v = \pm 4$

$$16 = 4\left(A^2 - 1^2\right)$$

$$4 = A^2 - 1$$

$$A^2 = 5$$

$$A = \pm\sqrt{5}$$



Distance travelled

$= \left[\left(-1+\sqrt{5}\right) - \left(-1-\sqrt{5}\right)\right] \times 2$

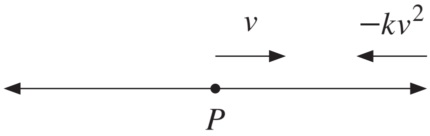
$= 2\sqrt{5} \times 2$

$= 4\sqrt{5}$ metres

Question 13 (c) (i)

Criteria	Marks
• Provides correct proof	3
• Integrates the equation of motion, or equivalent merit	2
• Obtains equation of motion, or equivalent merit	1

Sample answer:



$$m\ddot{x} = -kv^2$$

$$m = 1, \quad \ddot{x} = -kv^2$$

$$v\frac{dv}{dx} = -kv^2$$

$$\int \frac{dv}{v} = \int -k\,dx$$

$$\ln v = -kx + C \quad \text{since} \quad v \neq 0$$

When $t = 0$, $x = 0$ and $v = 40$

$$\therefore \quad \ln 40 = 0 + C \qquad C = \ln 40$$

$$\ln v - \ln 40 = -kx$$

$$\ln\left(\frac{v}{40}\right) = -kx$$

$$\therefore \quad v = 40e^{-kx}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$x = 15, \quad v = 10$$

$$10 = 40e^{-k15}$$

$$\frac{1}{4} = e^{-15k}$$

$$-15k = -\ln 4$$

$$k = \frac{\ln 4}{15}$$

Question 13 (c) (iii)

Criteria	Marks
• Provides correct solution	3
• Integrates equation of motion with respect to time, or equivalent merit	2
• Considers equation of motion with respect to time, or equivalent merit	1

Sample answer:

$$\ddot{x} = -kv^2$$

$$\frac{dv}{dt} = -kv^2$$

$$\int \frac{dv}{v^2} = \int -k\,dt$$

$$\frac{v^{-1}}{-1} = -kt + C$$

$$-\frac{1}{v} = -kt + C$$

When $t = 0 \quad v = 40$

$$-\frac{1}{40} = C$$

$$\therefore \quad -\frac{1}{v} = -kt - \frac{1}{40}$$

When will $v = 30$?

$$kt = \frac{1}{30} - \frac{1}{40} = \frac{1}{120}$$

$$t = \frac{1}{120} \times \frac{15}{\ln 4}$$

$$= \frac{1}{8\ln 4}$$

$$(\doteqdot 0.09 \text{ seconds})$$

Question 13 (d)

Criteria	Marks
• Provides correct proof	3
• Correctly applies the AM-GM inequality to $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, or equivalent merit	2
• Attempts to apply the AM-GM inequality to a relevant expression, or equivalent merit	1

Sample answer:

$$x+y \geq 2\sqrt{xy} \quad \Rightarrow \quad \sqrt{xy} \leq \frac{1}{2}(x+y) \qquad ①$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$$

$$\therefore \quad bc+ac+ab=abc$$

Now $a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab}$

$$\leq a \cdot \frac{1}{2}(b+c)+b \cdot \frac{1}{2}(a+c)+c \cdot \frac{1}{2}(a+b),$$ Using $AM-GM$ inequality from ①

$$\leq \frac{1}{2}[ab+ac+ab+bc+ac+bc]$$

$$\leq \frac{1}{\not{2}}[\not{2}(ab+ac+bc)]$$

$$\leq ab+ac+bc$$

$$\leq abc$$

Question 14 (a)

Criteria	Marks
• Provides correct proof	2
• Shows that $a^2 - 1$ is divisible by 4, or equivalent merit	1

Sample answer:

a is odd so let $a = 2p + 1$ where p is an integer

$$\begin{aligned} a^2 - 1 &= (2p+1)^2 - 1 \\ &= 4p^2 + 4p + 1 - 1 \\ &= 4p^2 + 4p \\ &= 4p(p+1) \end{aligned}$$

One of p and $p + 1$ will be even.

<u>Case 1</u> $p = 2q$ is even, where q is an integer

then $a^2 - 1 = 4 \times 2q(p+1)$

$= 8q(p+1)$

<u>Case 2</u> $p + 1 = 2r$ is even, where r is an integer

then $a^2 - 1 = 4p \times 2r$

$= 8p\text{r}$

Both of which are divisible by 8.

$\therefore$ If a is any odd integer, then $a^2 - 1$ is divisible by 8.

Question 14 (b)

Criteria	Marks
• Provides correct proof	3
• Proves that $p(k) \Rightarrow p(k+1)$, or equivalent merit	2
• Showing true for $n = 5$	1

Sample answer:

Prove: $\quad {}^{2n}C_n < 2^{2n-2} \qquad n \geq 5$

Initial case: $\quad n = 5 \qquad {}^{10}C_5 = 252 \qquad 2^8 = 256$

$$252 < 256$$

$\therefore$ True for $n = 5$.

Inductive step: Suppose statement is true for $n = k$

Assume $\quad {}^{2k}C_k < 2^{2k-2} \quad$ for $\quad k \geq 5$

ie $\quad \dfrac{(2k)!}{k! \cdot k!} < 2^{2k-2}$

Prove that $\quad {}^{2k+2}C_{k+1} < 2^{2k}$

$$\begin{aligned}
\text{LHS} &= {}^{2k+2}C_{k+1} \\
&= \frac{(2k+2)!}{(k+1)!\,(k+1)!} \\
&= \frac{(2k+2)(2k+1)(2k)!}{(k+1)^2 \cdot (k!)^2} \\
&= \frac{2(k+1)(2k+1)(2k)!}{(k+1)^2 \cdot (k)!^2} \\
&= \frac{2(2k+1)(2k)!}{(k+1)(k!)^2} \\
&= \frac{2(2k+1)}{(k+1)} \cdot {}^{2k}C_k \\
&< \frac{2(2k+1)}{(k+1)} \cdot 2^{2k-2} \qquad \text{using assumption } n = k \\
&= \frac{2k+1}{(k+1)} \cdot 2^{2k-1} \\
&< \frac{2k+2}{k+1} \cdot 2^{2k-1} \\
&= 2 \cdot 2^{2k-1} \\
&= 2^{2k} \\
&= \text{RHS}
\end{aligned}$$

${}^{2k+2}C_{k+1} < 2^{2k} \qquad$ as required

Hence, the statement is true by mathematical induction for all integers $n \geq 5$.

Question 14 (c)

Criteria	Marks
• Provides correct solution	2
• Recognises that $\frac{z}{w}$ is purely imaginary, or equivalent merit	1

Sample answer:

$\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$ z, w are complex numbers.

$\therefore \frac{z}{w}$ is purely imaginary.

$$\left|\frac{z-w}{z+w}\right| = \left|\frac{\frac{z}{w}-1}{\frac{z}{w}+1}\right|$$

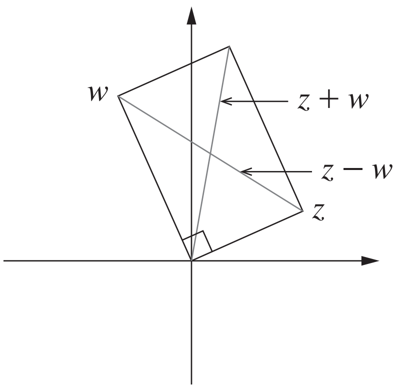
$\therefore$ Let $\frac{z}{w} = ki$

$$= \left|\frac{ki-1}{ki+1}\right|$$

$$= \frac{\sqrt{k^2+1}}{\sqrt{k^2+1}}$$

$$= 1$$

Answers could include:



$z - w$ and $z + w$ are the diagonals of a rectangle.

$\therefore |z-w| = |z+w|$

Hence $\left|\frac{z-w}{z+w}\right| = 1$

Question 14 (d)

Criteria	Marks
• Provides correct explanation	2
• Recognises that constants of integration are involved, or equivalent merit	1

Sample answer:

Using product rule

$$\frac{d(uv)}{dx} = u'v + uv'$$

$$u'v = \frac{d(uv)}{dx} - uv'$$

$$\int u'v\,dx = \int \frac{d(uv)}{dx}\,dx - \int uv'\,dx$$

But $\int \frac{d(uv)}{dx}\,dx = uv + C$ not just uv

So should have

$$\int \frac{1}{x}\,dx = 1 + C + \int \frac{1}{x}\,dx$$

$$0 = 1 + C \quad \text{for some constant } C$$

Answers could include:

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx$$

involves two indefinite integrals, and so two different constants

$$\ln|x| + C_1 = 1 + \ln|x| + C_2$$

$$C_1 = 1 + C_2 \quad \text{for some constants } C_1 \text{ and } C_2$$

Question 14 (e) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains $\overrightarrow{OR}$ in terms of $\underset{\sim}{a}$ and $\overrightarrow{PQ}$, or equivalent merit	1

Sample answer:

$\overrightarrow{OP} = \frac{3}{5}\underset{\sim}{a} \qquad \overrightarrow{PR} = k\overrightarrow{PQ} \qquad \overrightarrow{OQ} = 3\underset{\sim}{b}$

$$\begin{aligned}
\overrightarrow{OR} &= \overrightarrow{OP} + \overrightarrow{PR} \\
&= \frac{3}{5}\underset{\sim}{a} + k\overrightarrow{PQ} \\
&= \frac{3}{5}\underset{\sim}{a} + k\left(\frac{-3}{5}\underset{\sim}{a} + 3\underset{\sim}{b}\right) \\
&= \frac{3}{5}(1-k)\underset{\sim}{a} + 3k\underset{\sim}{b}
\end{aligned}$$

Question 14 (e) (ii)

Criteria	Marks
• Provides correct solution	2
• Equates coefficients for $\underset{\sim}{a}$ or $\underset{\sim}{b}$, or equivalent merit	1

Sample answer:

Given $\overrightarrow{OR} = (1-h)\underset{\sim}{a} + h\underset{\sim}{b}$

And, from part (i) $\overrightarrow{OR} = \frac{3}{5}(1-k)\underset{\sim}{a} + 3k\underset{\sim}{b}$

Equating coefficients

$$1 - h = \frac{3}{5}(1-k) \text{ and } 3k = h$$

$$1 - 3k = \frac{3}{5}(1-k)$$

$$5 - 15k = 3 - 3k$$

$$2 = 12k$$

$$k = \frac{1}{6}$$

Question 14 (e) (iii)

Criteria	Marks
• Provides correct solution	2
• Obtains a correct expression for $\overrightarrow{OT}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, or equivalent merit	1

Sample answer:

Using part (ii), $h = 3k$

$$\therefore \quad h = 3\left(\frac{1}{6}\right) = \frac{1}{2}$$

$$\overrightarrow{OT} = \lambda \overrightarrow{OR} = \lambda\left(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\right)$$

$$= \frac{\lambda}{2}\underset{\sim}{a} + \frac{\lambda}{2}\underset{\sim}{b}$$

$$\overrightarrow{OT} = \overrightarrow{OA} + \mu\overrightarrow{AQ} = \underset{\sim}{a} + \mu\left(-\underset{\sim}{a} + 3\underset{\sim}{b}\right)$$

$$= (1-\mu)\underset{\sim}{a} + 3\mu\underset{\sim}{b}$$

Equating coefficients of $\underset{\sim}{b}$

$$\frac{\lambda}{2} = 3\mu$$

$$\therefore \quad \lambda = 6\mu$$

Equating coefficients of $\underset{\sim}{a}$

$$\frac{\lambda}{2} = 1 - \mu$$

$$\lambda = 2 - 2\mu$$

$$\therefore \quad 6\mu = 2 - 2\mu$$

$$8\mu = 2$$

$$\mu = \frac{1}{4} \quad \text{and} \quad \lambda = \frac{3}{2}$$

$$\therefore \quad \overrightarrow{OT} = \frac{3}{2}\overrightarrow{OR}$$

$$= \frac{3}{4}\underset{\sim}{a} + \frac{3}{4}\underset{\sim}{b}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Line ℓ through A and B has equation $\overrightarrow{OX} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$ for points X on the line.

$$\begin{aligned}
\overrightarrow{OM} &= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} \\
&= \left(\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{a}\right) + \frac{1}{2}\underset{\sim}{b} \\
&= \overrightarrow{OA} + \frac{1}{2}\left(-\underset{\sim}{a} + \underset{\sim}{b}\right) \\
&= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}
\end{aligned}$$

$\therefore$ M lies on ℓ.

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains $\overrightarrow{OG}$ in terms of $\overrightarrow{OC}$ and $\overrightarrow{CM}$, or equivalent merit	1

Sample answer:

Line k through M and C has equation

$$\overrightarrow{OX} = \overrightarrow{OC} + \mu \overrightarrow{CM}, \text{ for points } X \text{ on the line.}$$

$$\begin{aligned}
\overrightarrow{OG} &= \frac{1}{3}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right) \\
&= \left(\underset{\sim}{c} - \frac{2}{3}\underset{\sim}{c}\right) + \frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b} \\
&= \underset{\sim}{c} + \frac{2}{3}\left(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{c}\right) \\
&= \overrightarrow{OC} + \frac{2}{3}\overrightarrow{CM}
\end{aligned}$$

$\therefore$ G lies on line k.

$\overrightarrow{OG} \neq \overrightarrow{OC}$ and $\overrightarrow{OG} \neq \overrightarrow{OM}$

So G lies between C and M.

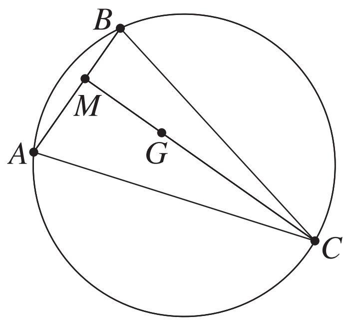
Question 15 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Shows that G lies within the unit circle, or equivalent merit	1

Sample answer:

The three points are on the unit circle.

$\therefore \quad |z| = 1$



The point equivalent to G, $\frac{1}{3}(x + w + z)$, lies on line CM and so is inside the unit circle.

And hence, $\left|\frac{1}{3}(x + w + z)\right| < 1$.

But, $|xwz| = 1$ and so all cube roots have modulus 1.

Therefore, $\frac{1}{3}(x + w + z)$ can't be a cube root of xwz.

Question 15 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly uses integration by parts to obtain a single integral term, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

$$I_n = \int_0^a x^{n+\frac{1}{2}} (a-x)^{\frac{1}{2}} \, dx$$

Using integration by parts,

$$u = x^{n+\frac{1}{2}} \qquad v' = (a-x)^{\frac{1}{2}}$$

$$u' = \left(n + \frac{1}{2}\right) x^{n-\frac{1}{2}} \qquad v = \frac{(a-x)^{\frac{3}{2}}}{-\frac{3}{2}}$$

$$I_n = \left[x^{n+\frac{1}{2}} \left(-\frac{2}{3}\right) (a-x)^{\frac{3}{2}} \right]_0^a + \frac{2}{3}\left(n+\frac{1}{2}\right) \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{3}{2}} \, dx$$

$$= \frac{2}{3}\left(n+\frac{1}{2}\right) \int_0^a x^{n-\frac{1}{2}} (a-x)^{\frac{3}{2}} \, dx$$

$$= \frac{1}{3}(2n+1) \int_0^a x^{n-\frac{1}{2}} (a-x)^1 (a-x)^{\frac{1}{2}} \, dx$$

$$= \frac{1}{3}(2n+1) \int_0^a a x^{n-\frac{1}{2}} (a-x)^{\frac{1}{2}} - x^{n+\frac{1}{2}} (a-x)^{\frac{1}{2}} \, dx$$

$$= \frac{1}{3}(2n+1)\left(aI_{n-1} - I_n\right)$$

$$I_n + \frac{1}{3}(2n+1) I_n = \frac{a}{3}(2n+1) I_{n-1}$$

$$\frac{(3+2n+1)}{3} I_n = \frac{a}{3}(2n+1) I_{n-1}$$

$$(2n+4) I_n = a(2n+1) I_{n-1}$$

Question 15 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly integrates the equation of motion, or equivalent merit	1

Sample answer:

$$a = \frac{27g}{x^3} - g \qquad t = 0 \quad v = 0 \quad x = 6$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{27g}{x^3} - g$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 27gx^{-3} - g$$

$$\frac{1}{2}v^2 = \frac{27gx^{-2}}{-2} - gx + C$$

$$\frac{1}{2}v^2 = \frac{-27g}{2x^2} - gx + C$$

$$v^2 = \frac{-27g}{x^2} - 2gx + D \qquad (D = 2C)$$

$$x = 6 \quad v = 0$$

$$0 = \frac{-27g}{36} - 12g + D$$

$$D = \frac{3}{4}g + 12g$$

$$= \frac{51}{4}g$$

$$v^2 = \frac{-27g}{x^2} - 2gx + \frac{51}{4}g$$

$$= g\left(\frac{51}{4} - 2x - \frac{27}{x^2}\right)$$

Question 15 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains a correct quadratic equation, or equivalent merit	1

Sample answer:

$$\frac{51}{4} - 2x - \frac{27}{x^2} = 0$$

$$51x^2 - 8x^3 - 108 = 0$$

$$8x^3 - 51x^2 + 108 = 0$$

Since $x = 6$ is a solution, $x - 6$ is a factor.

$$(x-6)\left(8x^2 + bx - 18\right) = 0$$

Equating terms in x^2,

$$b - 48 = -51$$

$$b = -3$$

$$(x-6)\left(8x^2 - 3x - 18\right) = 0$$

$$x = \frac{3 \pm \sqrt{9 - 4(8)(-18)}}{16}$$

$$x = \frac{3 \pm \sqrt{585}}{16}$$

$x \approx 1.699$ or -1.324

Since $x > 0$

$$x \approx 1.7$$

Question 15 (d)

Criteria	Marks
• Provides correct solution in terms of x	3
• Provides correct integral in terms of θ, or equivalent merit	2
• Completes the square OR • Attempts to substitute, or equivalent merit	1

Sample answer:

Let $1 - x = \sin\theta$

$$-dx = \cos\theta\, d\theta$$

$$dx = -\cos\theta\, d\theta$$

$$x = 1 - \sin\theta$$

$$\int \frac{2x^2}{\sqrt{2x - x^2}}\, dx = \int \frac{2x^2}{\sqrt{1 - \left(1 - 2x + x^2\right)}}\, dx$$

$$= \int \frac{2x^2}{\sqrt{1 - (1 - x)^2}}\, dx$$

$$= \int \frac{2(1 - \sin\theta)^2}{\sqrt{1 - \sin^2\theta}} (-\cos\theta)\, d\theta$$

$$= -2\int \frac{1 - 2\sin\theta + \sin^2\theta}{\cos\theta} (\cos\theta)\, d\theta$$

$$= -2\int 1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\, d\theta$$

$$= \int -2 + 4\sin\theta - 1 + \cos 2\theta\, d\theta$$

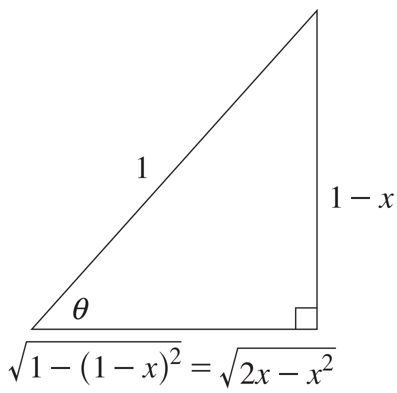
$$= -3\theta - 4\cos\theta + \frac{1}{2}\sin 2\theta + C$$

$$= -3\sin^{-1}(1 - x) - 4\sqrt{2x - x^2} + \frac{1}{2} 2\sin\theta\cos\theta + C$$

$$= -3\sin^{-1}(1 - x) - 4\sqrt{2x - x^2} + (1 - x)\sqrt{2x - x^2} + C$$

$$= -3\sin^{-1}(1 - x) - 3\sqrt{2x - x^2} - x\sqrt{2x - x^2} + C$$

$$= -3\sin^{-1}(1 - x) - (x + 3)\sqrt{2x - x^2} + C$$



Question 15 (d) (continued)

Answers could include:

Using $\cos\theta = 1 - x$ as substitution

$$\int \frac{2x^2}{\sqrt{2x - x^2}}\,dx = 2\int \frac{x^2}{\sqrt{1 - (1-x)^2}}\,dx$$

$$= 2\int \frac{(1-\cos\theta)^2}{\sqrt{1-\cos^2\theta}}\sin\theta\, d\theta$$

$$= 2\int \frac{1 - 2\cos\theta + \cos^2\theta}{\sin\theta}\sin\theta\, d\theta$$

$$= 2\int 1 - 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\, d\theta$$

$$= \int 2 - 4\cos\theta + 1 + \cos 2\theta\, d\theta$$

$$= \int 3 - 4\cos\theta + \cos 2\theta\, d\theta$$

$$= 3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta + D$$

$$= 3\cos^{-1}(1-x) - 4\sqrt{2x - x^2} + \sqrt{2x-x^2}\,(1-x) + D$$

$$= 3\cos^{-1}(1-x) - (x+3)\sqrt{2x - x^2} + D$$

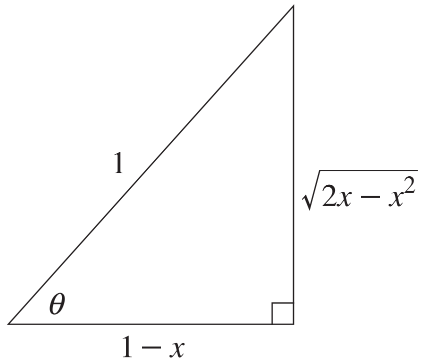
$2x - x^2 = 1 - (1-x)^2$

Let $\cos\theta = 1 - x$

$x = 1 - \cos\theta$

$\frac{dx}{d\theta} = \sin\theta$

$dx = \sin\theta\, d\theta$



Question 16 (a)

Criteria	Marks
• Provides correct solution	4
• Attempts to use a relevant trigonometric fact, with $2a = k\sin 2ka$ to obtain result, or equivalent merit	3
• Obtains a correct equation involving $\sin 2ka$, or equivalent merit	2
• Provides a correct expression using the slope of OP, or equivalent merit	1

Sample answer:

At $P(a, b)$ the slope of the curve is $-k\sin ka$.

Since OP is perpendicular to the tangent at P

$$\frac{b}{a} \times (-k\sin ka) = -1$$

$$-\frac{a}{b} = -k\sin ka$$

But $b = \cos ka$

$$a = k\sin ka \cos ka$$
$$2a = k\sin 2ka$$

Let $X = 2a$, then we have

$$X = k\sin kX$$

If $k \le 1$, then, as $X > 0$

$$k\sin kX \le \sin kX < kX \le X$$

But $X = k\sin kX$

So by contradiction $k > 1$

Question 16 (b) (i)

Criteria	Marks
• Provides correct solution	3
• Obtains $(\gamma + \overline{\gamma})^3 - 3(\gamma + \overline{\gamma}) + 1 = \gamma^3 + \overline{\gamma}^3 + 1$, or equivalent merit	2
• Substitutes $\gamma + \overline{\gamma}$ into $z^3 - 3z + 1 = 0$, or equivalent merit	1

Sample answer:

$w = e^{\frac{2\pi i}{3}}$ as $|\gamma| = |\overline{\gamma}| = 1$, $\gamma = e^{i\theta}$, $\overline{\gamma} = e^{-i\theta}$, $\gamma\overline{\gamma} = 1$

$$
\begin{aligned}
(\gamma + \overline{\gamma})^3 - 3(\gamma + \overline{\gamma}) + 1 &= \gamma^3 + 3\gamma^2\overline{\gamma} + 3\gamma\overline{\gamma}^2 + \overline{\gamma}^3 - 3\gamma - 3\overline{\gamma} + 1 \\
&= \gamma^3 + \overline{\gamma}^3 + 3\gamma + 3\overline{\gamma} - 3\gamma - 3\overline{\gamma} + 1 \qquad \text{as } \gamma\overline{\gamma} = 1 \\
&= \gamma^3 + \overline{\gamma}^3 + 1 \qquad \gamma^3 = e^{\frac{2\pi i}{3}} \\
&= 2\cos\frac{2\pi}{3} + 1 \qquad \overline{\gamma}^3 = e^{-\frac{2\pi i}{3}} \\
&= 2\left(-\frac{1}{2}\right) + 1 \\
&= 0
\end{aligned}
$$

$\therefore$ $\gamma + \overline{\gamma}$ is a real root of $z^3 - 3z + 1 = 0$.

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Attempts to use inductive reasoning, or equivalent merit	2
• Shows $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{1}{8}$, or equivalent merit	1

Sample answer:

The cube roots of $w = e^{\frac{2\pi i}{3}}$ are $e^{\frac{1}{3}\left(\frac{2\pi i}{3}\right)}$, $e^{\frac{1}{3}\left(\frac{2\pi}{3}+2\pi\right)i}$ and $e^{\frac{1}{3}\left(\frac{2\pi}{3}+4\pi\right)i}$ that is $e^{\frac{2\pi i}{9}}$, $e^{\frac{8\pi i}{9}}$ and $e^{\frac{14\pi i}{9}}$. By part (i) each of these plus its conjugate will be a root of $z^3 - 3z + 1 = 0$.

So $2\cos\frac{2\pi}{9}$, $2\cos\frac{8\pi}{9}$ and $2\cos\frac{14\pi}{9}$ are roots of $z^3 - 3z + 1 = 0$.

Noting that
$$\begin{aligned}\cos\frac{14\pi}{9} &= \cos\left(-\frac{14\pi}{9}\right)\\ &= \cos\left(2\pi - \frac{14\pi}{9}\right)\\ &= \cos\frac{4\pi}{9}\end{aligned}$$

The roots of $z^3 - 3z + 1 = 0$ are $2\cos\frac{2\pi}{9}$, $2\cos\frac{4\pi}{9}$ and $2\cos\frac{8\pi}{9}$.
(These are different as $\cos\theta$ is decreasing for $0 < \theta < \pi$.)
The product of the roots is -1.

Hence
$$\begin{aligned}8\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} &= -1\\ \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} &= -\frac{1}{8}\end{aligned}$$

This is the base case of this induction.

Noting that
$$\begin{aligned}\cos\frac{2^{k+3}\pi}{9} &= \cos\left(\frac{8}{9}\times 2^k\pi\right)\\ &= \cos\left(-\frac{8}{9}\times 2^k\pi\right), \text{ cos even}\\ &= \cos\left(2^k\pi - \frac{8}{9}2^k\pi\right), \text{ for } k \geq 1\\ &= \cos\frac{2^k\pi}{9}\end{aligned}$$

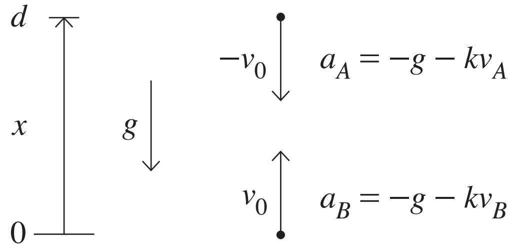
Hence $\cos\frac{2^{k+1}\pi}{9}\cos\frac{2^{k+2}\pi}{9}\cos\frac{2^{k+3}\pi}{9} = \cos\frac{2^k\pi}{9}\cos\frac{2^{k+1}\pi}{9}\cos\frac{2^{k+2}\pi}{9}$ (verifying the induction step).

So $\cos\frac{2^n\pi}{9}\cos\frac{2^{n+1}\pi}{9}\cos\frac{2^{n+2}\pi}{9} = -\frac{1}{8}$ for all integers $n \geq 1$.

Question 16 (c)

Criteria	Marks
• Provides correct solution	4
• Finds position of either particle or $\ell = x_A - x_B$ in terms of t, or equivalent merit	3
• Finds velocity of either particle or $\frac{d\ell}{dt}$, or equivalent merit	2
• Obtains correct equations of motion, or equivalent merit	1

Sample answer:



Let $\ell = x_A - x_B$

$$\frac{d^2\ell}{dt^2} = a_A - a_B$$

$$= \cancel{-g} - kv_A \cancel{+g} + kv_B$$

$$= -k\left(v_A - v_B\right)$$

$$= -k\left(\frac{dx_A}{dt} - \frac{dx_B}{dt}\right)$$

$$= -k\frac{d}{dt}\left(x_A - x_B\right)$$

$$= -k\frac{d\ell}{dt}$$

$$\frac{d\ell}{dt} = Le^{-kt}, \text{ where } L \text{ is a constant}$$

$$\left[t = 0, \quad \frac{d\ell}{dt} = -2v_0\right]$$

$$\frac{d\ell}{dt} = -2v_0 e^{-kt}$$

$$\ell = \frac{2v_0}{k}e^{-kt} + C$$

$$\left[t = 0, \quad \ell = d\right]$$

$$d = \frac{2v_0}{k} + C$$

$$C = \frac{kd - 2v_0}{k}$$

$$\ell = \frac{2v_0}{k}e^{-kt} + \frac{kd - 2v_0}{k}$$

$$\left[\ell = 0\right]$$

$$2v_0 e^{-kt} + kd - 2v_0 = 0$$

$$\frac{2v_0 - kd}{2v_0} = e^{-kt}$$

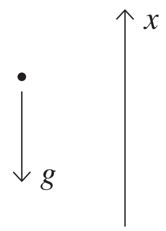
$$\frac{2v_0}{2v_0 - kd} = e^{kt}$$

$$t = \frac{1}{k}\log\left(\frac{2v_0}{2v_0 - kd}\right)$$

Question 16 (c) (continued)

Answers could include:

The equation of motion for both particles is the same.



$$\frac{dv}{dt} = -(g + kv)$$

$$\int \frac{dv}{g + kv} = \int -dt$$

$$\frac{1}{k}\log\left| g + kv \right| = -t + C_1$$

$g + kV$ can never be 0, so can't change sign.

For A, v is initially $-v_0 > -\frac{g}{k}$

So $v > -\frac{g}{k}$

Hence $g + kv > 0$

For B, v is $v_0 > 0$

So $g + kv_0 > 0$

Hence $g + kv > 0$ for all t.

So, in both cases, $g + kv > 0$ for all t.

$$\frac{1}{k}\log(g + kv) = -t + C_1$$

$$g + kv = L_1 e^{-kt}$$

$$v = -\frac{g}{k} + L_1 e^{-kt}$$

Question 16 (c), answers could include (continued)

For Particle A,

When $t = 0$, $v_A = -v_0$

$$-v_0 = -\frac{g}{k} + \frac{L_1}{k}$$

$$\frac{L_1}{k} = \frac{g}{k} - v_0$$

$$v_A = -\frac{g}{k} + \left(\frac{g}{k} - v_0\right)e^{-kt}$$

$$x_A = -\frac{gt}{k} - \frac{1}{k}\left(\frac{g}{k} - v_0\right)e^{-kt} + L_2$$

When $t = 0$, $x_A = d$

$$d = -\frac{1}{k}\left(\frac{g}{k} - v_0\right) + L_2$$

$$L_2 = d + \frac{1}{k}\left(\frac{g}{k} - v_0\right)$$

$$x_A = -\frac{gt}{k} - \frac{1}{k}\left(\frac{g}{k} - v_0\right)e^{-kt} + d + \frac{1}{k}\left(\frac{g}{k} - v_0\right)$$

For Particle B,

When $t = 0$, $v_B = v_0$

$$v_0 = -\frac{g}{k} + \frac{L_3}{k}$$

$$v_B = -\frac{g}{k} + \left(\frac{g}{k} + v_0\right)e^{-kt}$$

$$x_B = -\frac{gt}{k} - \frac{1}{k}\left(\frac{g}{k} + v_0\right)e^{-kt} + L_4$$

When $t = 0$, $x_B = 0$

$$0 = -\frac{1}{k}\left(\frac{g}{k} + v_0\right) + L_4$$

$$x_B = -\frac{gt}{k} - \frac{1}{k}\left(\frac{g}{k} + v_0\right)e^{-kt} + \frac{1}{k}\left(\frac{g}{k} + v_0\right)$$

Particles meet when $x_A = x_B$

$$-\frac{gt}{k} - \frac{1}{k}\left(\frac{g}{k} - v_0\right)e^{-kt} + d + \frac{1}{k}\left(\frac{g}{k} - v_0\right) = -\frac{gt}{k} - \frac{1}{k}\left(\frac{g}{k} + v_0\right)e^{-kt} + \frac{1}{k}\left(\frac{g}{k} + v_0\right)$$

$$\frac{v_0}{k}e^{-kt} + d - \frac{v_0}{k} = -\frac{v_0}{k}e^{-kt} + \frac{v_0}{k}$$

$$\frac{2v_0}{k}e^{-kt} = \frac{2v_0}{k} - d$$

$$e^{-kt} = \frac{2v_0 - kd}{2v_0} > 0$$

$$t = \frac{1}{k}\log\left(\frac{2v_0}{2v_0 - kd}\right)$$

2024 HSC Mathematics Extension 2

Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-V1 Further Work with Vectors	MEX12-3
2	1	MEX-P1 The Nature of Proof	MEX12-8
3	1	MEX-P1 The Nature of Proof	MEX12-2
4	1	MEX-N2 Using Complex Numbers	MEX12-4
5	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
6	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
7	1	MEX-N2 Using Complex Numbers	MEX12-4
8	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
9	1	MEX-N2 Using Complex Numbers	MEX12-4
10	1	MEX-V1 Further Work with Vectors	MEX12-3

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	MEX-C1 Further Integration	MEX12-5
11 (b) (i)	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
11 (b) (ii)	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
11 (c)	2	MEX-V1 Further Work with Vectors	MEX12-3
11 (d)	3	MEX-C1 Further Integration	MEX12-5
11 (e) (i)	2	MEX-N1 Introduction to Complex Numbers	MEX12-4
11 (e) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12-4
11 (f)	3	MEX-N2 Using Complex Numbers	MEX12-4
12 (a) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3
12 (a) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3
12 (b)	3	MEX-C1 Further Integration	MEX12-5
12 (c) (i)	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
12 (c) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12-4
12 (d)	2	MEX-P1 The Nature of Proof	MEX12-8
12 (e) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3
12 (e) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3
13 (a) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3
13 (a) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3
13 (b)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
13 (c) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
13 (c) (ii)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
13 (c) (iii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
13 (d)	3	MEX-P1 The Nature of Proof	MEX12-8

Question	Marks	Content	Syllabus outcomes
14 (a)	2	MEX-P1 The Nature of Proof	MEX12-8
14 (b)	3	MEX-P2 Further Proof by Mathematical Induction	MEX12-8
14 (c)	2	MEX-N2 Using Complex Numbers	MEX12-4
14 (d)	2	MEX-P1 The Nature of Proof	MEX12-2
14 (e) (i)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (e) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (e) (iii)	2	MEX-V1 Further Work with Vectors	MEX12-3
15 (a) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3
15 (a) (ii)	2	MEX-V1 Further Work with Vectors	MEX12-3
15 (a) (iii)	2	MEX-V1 Further Work with Vectors	MEX12-3
15 (b)	3	MEX-C1 Further Integration	MEX12-5
15 (c) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (c) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (d)	3	MEX-C1 Further Integration	MEX12-5
16 (a)	4	MEX-P1 The Nature of Proof	MEX12-1, MEX12-8
16 (b) (i)	3	MEX-N2 Using Complex Numbers	MEX12-4
16 (b) (ii)	3	MEX-N2 Using Complex Numbers	MEX12-4
16 (c)	4	MEX-M1 Applications of Calculus to Mechanics	MEX12-6