2023 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	D
3	B
4	C
5	A
6	B
7	A
8	D
9	D
10	B

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Completes the square OR • Uses quadratic formula	1

Sample answer:

$$\begin{aligned} z &= \frac{3 \pm \sqrt{9-16}}{2} \\ &= \frac{3 \pm \sqrt{-7}}{2} \\ &= \frac{3}{2} \pm \frac{\sqrt{7}}{2} i \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly evaluates $\underset{\sim}{a} \cdot \underset{\sim}{b}$ and $\lvert \underset{\sim}{a} \rvert$ and $\lvert \underset{\sim}{b} \rvert$, or equivalent merit	2
• Evaluates $\underset{\sim}{a} \cdot \underset{\sim}{b}$ or $\lvert \underset{\sim}{a} \rvert$ or $\lvert \underset{\sim}{b} \rvert$, or equivalent merit	1

Sample answer:

$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= \big(1 \times (-1)\big) + (2 \times 4) + (-3 \times 2) \\ &= -1 + 8 - 6 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lvert \underset{\sim}{a} \rvert &= \sqrt{1^2 + 2^2 + (-3)^2} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} \lvert \underset{\sim}{b} \rvert &= \sqrt{(-1)^2 + 4^2 + 2^2} \\ &= \sqrt{21} \end{aligned}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert \underset{\sim}{a} \rvert \lvert \underset{\sim}{b} \rvert \cos\theta$$

$$\therefore \qquad \cos\theta = \frac{1}{\sqrt{14} \cdot \sqrt{21}}$$

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{1}{\sqrt{14} \cdot \sqrt{21}}\right) \\ &= 86.656\ldots \\ &\approx 87^\circ \qquad \text{(nearest degree)} \end{aligned}$$

Question 11 (c)

Criteria	Marks
• Provides a correct equation	2
• Finds $\overline{AB}$, or equivalent merit	1

Sample answer:

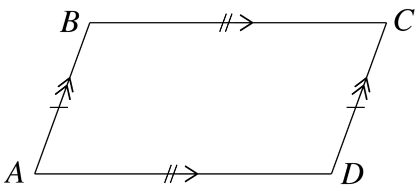
$$
\begin{aligned}
\overline{AB} &= \overline{AO} + \overline{OB} \\
&= \overrightarrow{OB} + \overrightarrow{OA} \\
&= \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}
\end{aligned}
$$

$\therefore$ A vector equation of the line is $\underset{\sim}{r} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R}$

Question 11 (d)

Criteria	Marks
• Provides correct proof	2
• Explains why $\overline{AB} = \overline{DC}$, or equivalent merit	1

Sample answer:

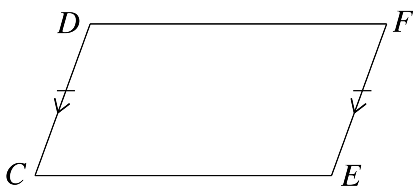


$\overline{AB} = \overline{DC}$ (opposite sides of a parallelogram are equal in length and parallel)

But in parallelogram *ABEF*

$\overline{AB} = \overline{FE}$ (opposite sides of a parallelogram are equal in length and parallel)

$\therefore \quad \overline{DC} = \overline{FE}$



$\therefore \quad CDFE$ is a parallelogram (one pair of opposite sides both equal in length and parallel)

Question 11 (e)

Criteria	Marks
• Provides correct period and central point of motion	2
• Provides correct period or centre, or equivalent merit	1

Sample answer:

$$\ddot{x} = -3^2(x-4)$$

$\therefore \; n = 3$ and $c = 4$

$$\therefore \; T = \frac{2\pi}{n} = \frac{2\pi}{3}$$

$\therefore$ Period $= \dfrac{2\pi}{3}$ and centre is $x = 4$

Question 11 (f)

Criteria	Marks
• Provides correct solution	4
• Obtains the antiderivative, or equivalent merit	3
• Shows $\dfrac{5x-3}{(x+1)(x-3)} = \dfrac{2}{x+1} + \dfrac{3}{x-3}$, or equivalent merit	2
• Attempts to use partial fractions, or equivalent merit	1

Sample answer:

Let $\dfrac{5x-3}{(x+1)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x-3}$

$$\begin{aligned} \therefore \quad 5x-3 &= A(x-3)+B(x+1) \\ &= Ax-3A+Bx+B \end{aligned}$$

Equating coefficients

$$5 = A + B \qquad (1)$$
$$-3 = -3A + B \qquad (2)$$

(1) – (2): $8 = 4A$

$\therefore A = 2$

In (1) $B = 3$

$$\begin{aligned} \therefore \quad \int_0^2 \frac{5x-3}{(x+1)(x-3)}\,dx &= \int_0^2 \frac{2}{x+1} + \frac{3}{x-3}\,dx \\ &= \Big[2\ln|x+1| + 3\ln|x-3|\Big]_0^2 \\ &= \big(2\ln 3 + 3\ln|-1|\big) - \big(2\ln 1 + 3\ln|-3|\big) \\ &= 2\ln 3 - 3\ln 3 \\ &= -\ln 3 \quad \text{or} \quad \ln\left(\frac{1}{3}\right) \end{aligned}$$

Question 12 (a)

Criteria	Marks
• Provides correct proof	3
• Shows that 23 is a factor of a relevant integer, or equivalent merit	2
• Attempts a proof by contradiction or writes $\sqrt{23} = \frac{p}{q}$ for integers p, q, or equivalent merit	1

Sample answer:

Assume the contradiction, that $\sqrt{23}$ is rational.

Let $\sqrt{23} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Also assume that p and q are coprime.

$$\left(\sqrt{23}\right)^2 = \left(\frac{p}{q}\right)^2$$

$$23 = \frac{p^2}{q^2}$$

$$23q^2 = p^2$$

23 divides LHS, so 23 divides p^2.

Since 23 is prime, 23 also divides p.

So $p = 23m$, for some $m \in \mathbb{Z}$

$$\therefore \quad 23q^2 = (23m)^2$$

$$q^2 = 23m^2$$

So similarly 23 also divides q.

But now p and q have a common factor of 23, contradicting the assumption. Therefore, the assumption is false and $\sqrt{23}$ is irrational.

Question 12 (b)

Criteria	Marks
• Provides correct proof	2
• Expands the numerator of the LHS and splits into two fractions, or equivalent merit	1

Sample answer:

$$\frac{(x+y)^2}{x^2+y^2} = \frac{x^2+2xy+y^2}{x^2+y^2}$$

$$= 1 + \frac{2xy}{x^2+y^2}$$

But $\dfrac{a+b}{2} \geq \sqrt{ab}$, therefore $\dfrac{x^2+y^2}{2} \geq \sqrt{x^2y^2}$

and $\quad x^2+y^2 \geq 2|xy| \geq 2xy \qquad$ (since $\sqrt{x^2} = |x|$ and here we have $a,\ b \geq 0$

$$1 \geq \frac{2xy}{x^2+y^2}$$

$$\therefore \quad \frac{(x+y)^2}{x^2+y^2} \leq 1+1$$

$$\leq 2$$

<u>Alternative solution</u>

$$(x-y)^2 \geq 0 \qquad \forall x,\ y \in \mathbb{R}$$

$$x^2 - 2xy + y^2 \geq 0$$

$$2x^2 - 2xy + 2y^2 \geq x^2 + y^2$$

$$2\left(x^2+y^2\right) \geq x^2+y^2+2xy$$

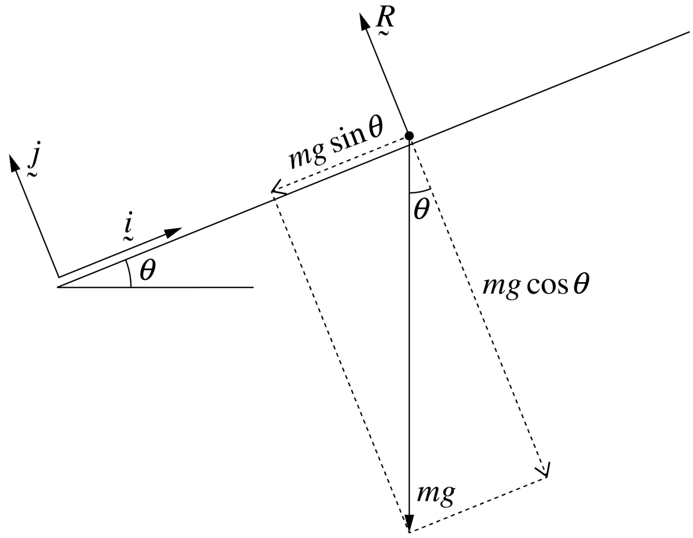
$$\geq (x+y)^2$$

$$\therefore \quad 2 \geq \frac{(x+y)^2}{x^2+y^2}$$

Question 12 (c) (i)

Criteria	Marks
• Provides correct proof	2
• Attempts to resolve a force into two perpendicular components or writes $\underset{\sim}{F} = \underset{\sim}{R} + m\underset{\sim}{g}$, or equivalent merit	1

Sample answer:



Resolving parallel and perpendicular to the slope gives

$$\underset{\sim}{R} - mg\cos\theta\,\underset{\sim}{j} = 0$$

and $\quad \underset{\sim}{F} = -(mg\sin\theta)\underset{\sim}{i} = 0$

$\therefore$ Resultant force is $\underset{\sim}{F} = -(mg\sin\theta)\underset{\sim}{i}$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds a correct equation for acceleration and attempts to integrate, or equivalent merit	1

Sample answer:

$$m\underset{\sim}{a} = -mg\sin\theta\,\underset{\sim}{i}$$
$$\underset{\sim}{a} = -g\sin\theta\,\underset{\sim}{i}$$
$$\frac{d\underset{\sim}{v}}{dt} = -g\sin\theta\,\underset{\sim}{i}$$
$$\underset{\sim}{v} = -gt\sin\theta\,\underset{\sim}{i} + C$$

But when $t = 0$ $\quad \underset{\sim}{v} = 0$
So $\underset{\sim}{v} = -gt\sin\theta\,\underset{\sim}{i}$

Question 12 (d)

Criteria	Marks
• Provides correct solution	3
• Finds one correct cube root, or equivalent merit	2
• Writes $2-2i$ in exponential form, or equivalent merit	1

Sample answer:

Let $z^3 = 2-2i$ where $z = re^{i\theta}$, $r \in \mathbb{R}$

$$\therefore\ r^3 e^{3i\theta} = 2-2i$$

But $2-2i = \sqrt{8}e^{-\frac{i\pi}{4}}$

$$\therefore\ r^3 = \sqrt{8} \quad \text{and} \quad 3\theta = -\frac{\pi}{4} \text{ or } \frac{7\pi}{4} \text{ or } \frac{15\pi}{4}$$
$$r = \sqrt{2} \quad \text{and} \quad \theta = -\frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{15\pi}{12}$$

$$\therefore\ z = \sqrt{2}e^{-\frac{i\pi}{12}} \text{ or } \sqrt{2}e^{\frac{7i\pi}{12}} \text{ or } \sqrt{2}e^{\frac{5i\pi}{4}}$$

Question 12 (e) (i)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

Since $P(z)$ has real coefficients, zeros occur in complex conjugate pairs

$\therefore \quad 2 + i$ a zero $\Rightarrow \overline{2+i} = 2 - i$ is also a zero

Question 12 (e) (ii)

Criteria	Marks
• Provides correct solution	2
• Writes down a correct equation involving the sum of the roots or the product of the roots, or attempts a long division of polynomials, or equivalent merit	1

Sample answer:

$$P(z) = z^4 - 3z^3 + cz^2 + dz - 30$$

Sum of roots $= -\dfrac{b}{a} = 3$

Product of roots $= \dfrac{e}{a} = -30$

Let other two zeros be α and β

$$\begin{aligned} \therefore \quad (2+i)+(2-i)+\alpha+\beta &= 3 \\ 4+\alpha+\beta &= 3 \\ \alpha+\beta &= -1 \qquad (1) \end{aligned}$$

$$\begin{aligned} (2+i)(2-i)\alpha\beta &= -30 \\ (4+1)\alpha\beta &= -30 \\ \alpha\beta &= -6 \qquad (2) \end{aligned}$$

$\therefore \quad$ Other zeros are $z = 2$ and $z = -3$

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Splits the integral into the sum of two integrals and correctly integrates one, or equivalent merit	2
• Completes the square for $5-4x-x^2$ or writes $1-x=\frac{1}{2}(-4-2x)+3$ or equivalent merit	1

Sample answer:

$$\int \frac{1-x}{\sqrt{5-4x-x^2}}\,dx$$

$$= \int \frac{-2-x}{\sqrt{5-4x-x^2}}\,dx + \int \frac{3}{\sqrt{5-4x-x^2}}\,dx$$

$$= \sqrt{5-4x-x^2} + \int \frac{3}{\sqrt{9-\left(4+4x+x^2\right)}}\,dx$$

$$= \sqrt{5-4x-x^2} + 3\sin^{-1}\left(\frac{x+2}{3}\right) + C$$

$$\frac{d}{dx}\sqrt{5-4x-x^2} = \frac{1}{2\sqrt{5-4x-x^2}} \times (-4-2x)$$

$$= \frac{-2-x}{\sqrt{5-4x-x^2}}$$

Question 13 (b) (i)

Criteria	Marks
• Provides correct proof	1

Sample answer:

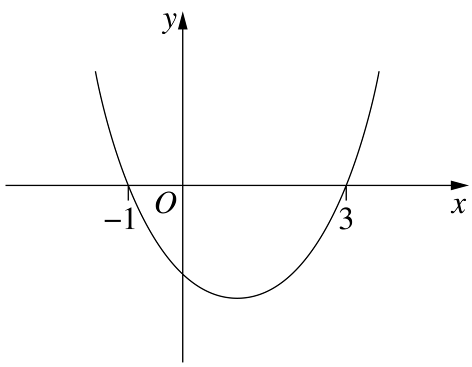
$$k^2 - 2k - 3 = k^2 - 2k + 1 - 4$$
$$= (k-1)^2 - 4$$

$$(k-1)^2 - 4 \geq 0 \quad \text{for all } k \geq 3$$
$$\therefore \; k^2 - 2k - 3 \geq 0 \quad \text{for all } k \geq 3$$

Alternative solution

$$k^2 - 2k - 3 = (k-3)(k+1)$$
$$\geq 0 \qquad \text{for } k \geq 3 \text{ or } k \leq -1$$

Since $k \geq 3$, we can ignore $k \leq -1$



Question 13 (b) (ii)

Criteria	Marks
• Provides correct proof	3
• Establishes the inductive step, or equivalent merit	2
• Establishes the base case, or equivalent merit	1

Sample answer:

Let $P(n)$ be the statement $2^n \geq n^2 - 2$

Consider $n = 3$ case

LHS $= 2^3 = 8$

RHS $= 3^2 - 2 = 9 - 2 = 7$

$8 \geq 7$ so $P(3)$ is true

Assume $P(k)$ is true for some $k \geq 3$

Thus, $2^k \geq k^2 - 2,\ k \geq 3$

Prove that $P(k + 1)$ is true.

ie $\quad 2^{k+1} \geq (k+1)^2 - 2$

$$2^{k+1} \geq k^2 + 2k - 1$$

$$\begin{aligned}
\text{LHS} &= 2^{k+1} \\
&= 2\left(2^k\right) \\
&\geq 2\left(k^2 - 2\right) && \text{since we assumed } P(k) \text{ is true} \\
&= 2k^2 - 4 \\
&= k^2 + 2k - 1 + k^2 - 2k - 3 \\
\therefore \text{LHS} &\geq k^2 + 2k - 1 && \text{using part (i)} \\
&= \text{RHS}
\end{aligned}$$

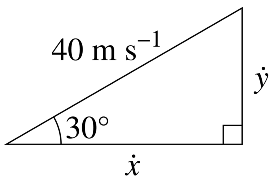
Therefore $P(k + 1)$ is true

Hence, using the principle of mathematical induction, $2^n \geq n^2 - 2$ for all integers $n \geq 3$.

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:



$$t = 0 \Rightarrow \quad \frac{\dot{x}}{40} = \cos 30^\circ$$
$$\dot{x} = 40\cos 30^\circ$$
$$= 20\sqrt{3}$$

$$t = 0 \Rightarrow \quad \frac{\dot{y}}{40} = \sin 30^\circ$$
$$\dot{y} = 40\sin 30^\circ$$
$$= 20$$

$$\therefore \quad \mathbf{v}(0) = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds one component of $\underset{\sim}{v}(t)$, or equivalent merit	2
• Provides a relevant force or acceleration equation, or equivalent merit	1

Sample answer:

The resulting force $\underset{\sim}{F} = -4\underset{\sim}{v} + m\underset{\sim}{g}$

$$\begin{aligned}\underset{\sim}{F} &= 1 \times \underset{\sim}{a}(t) \\ &= \underset{\sim}{a}(t)\end{aligned}$$

$$\underset{\sim}{a}(t) = \begin{pmatrix} -4\dot{x} \\ -4\dot{y} - 10 \end{pmatrix} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

$\ddot{x} = -4\dot{x}$

$$\begin{aligned}\frac{d\dot{x}}{dt} = -4\dot{x} \qquad &\Rightarrow \quad \frac{d\dot{x}}{\dot{x}} = -4dt \\ &\Rightarrow \quad \ln \dot{x} = -4t + C \\ &\Rightarrow \quad \dot{x} = Ae^{-4t} \\ &\qquad\ \ \dot{x} = 20\sqrt{3}e^{-4t} \qquad \text{since } \dot{x} = 20\sqrt{3} \text{ when } t = 0\end{aligned}$$

$$\begin{aligned}\ddot{y} &= -4\dot{y} - 10 \\ &= -4\left(\dot{y} - \frac{5}{2}\right)\end{aligned}$$

So $\dot{y} = -\frac{5}{2} + Be^{-4t}$

When $t = 0$ $\quad \dot{y} = 20$

$20 = -\frac{5}{2} + B$

$\frac{45}{2} = B$

$\therefore \quad \dot{y} = -\frac{5}{2} + \frac{45}{2}e^{-4t}$

$$\underset{\sim}{v}(t) = \begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{y} \end{pmatrix} = \begin{pmatrix} 20\sqrt{3}e^{-4t} \\ \frac{45}{2}e^{-4t} - \frac{5}{2} \end{pmatrix}$$

Question 13 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Finds one component of $\underset{\sim}{\mathbf{r}}(t)$, or equivalent merit	1

Sample answer:

By integrating $\mathbf{v}(t)$ with respect to t, we get

$$\mathbf{r}(t) = \begin{pmatrix} \dfrac{20\sqrt{3}}{-4}e^{-4t} + C_1 \\ \\ \dfrac{45}{-8}e^{-4t} - \dfrac{5}{2}t + C_2 \end{pmatrix} \qquad \text{Where } C_1 \text{ and } C_2 \text{ are constants}$$

$\mathbf{r}(0) = \mathbf{0}$ therefore

$-5\sqrt{3} + C_1 = 0 \quad \text{so} \quad C_1 = 5\sqrt{3}$

$-\dfrac{45}{8} + C_2 = 0 \quad \text{so} \quad C_2 = \dfrac{45}{8}$

Substituting back in $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \begin{pmatrix} -5\sqrt{3}\,e^{-4t} + 5\sqrt{3} \\ \\ -\dfrac{45}{8}e^{-4t} - \dfrac{5}{2}t + \dfrac{45}{8} \end{pmatrix} = \begin{pmatrix} 5\sqrt{3}\left(1 - e^{-4t}\right) \\ \\ \dfrac{45}{8}\left(1 - e^{-4t}\right) - \dfrac{5}{2}t \end{pmatrix}$$

as required.

Question 13 (c) (iv)

Criteria	Marks
• Provides correct answer	2
• Finds time of flight, or equivalent merit	1

Sample answer:

The particle lands on the ground at a time t which satisfies $y = 0$, that is,

$$\frac{45}{8}\left(1-e^{-4t}\right)-\frac{5}{2}t=0$$

$$\frac{45}{8}\left(1-e^{-4t}\right)=\frac{5}{2}t$$

$$\left(1-e^{-4t}\right)=\frac{5}{2}\times\frac{8}{45}t$$

$$1-e^{-4t}=\frac{4}{9}t$$

Solution $t \approx 2.25$ according to the diagram provided

$$\therefore \quad \text{range} = 5\sqrt{3}\left(1-e^{-4\times 2.25}\right)$$

$$= 8.65918\ldots$$

$$= 8.7 \text{ m} \qquad \text{(rounded to 1 decimal place)}$$

Question 14 (a) (i)

Criteria	Marks
• Provides correct proof	3
• Obtains $\lvert z+w \rvert^2 = \frac{1}{4}\left(\left(\sqrt{3}-\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right)^2\right)$, or equivalent merit	2
• Finds z or w in Cartesian form, or equivalent merit	1

Sample answer:

We have $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $w = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

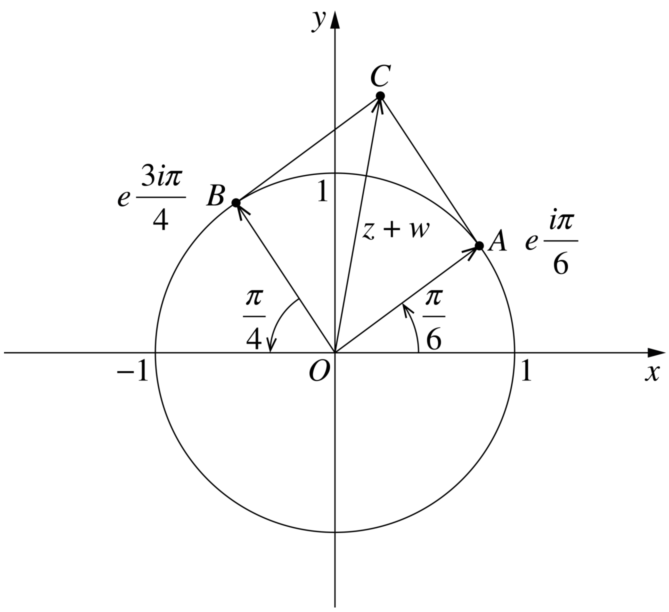
$$\begin{aligned} z+w &= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= \left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)i \\ &= \frac{\sqrt{3}-\sqrt{2}}{2} + \left(\frac{1+\sqrt{2}}{2}\right)i \end{aligned}$$

$$\begin{aligned} \lvert z+w \rvert^2 &= \frac{1}{4}\left[\left(\sqrt{3}-\sqrt{2}\right)^2 + \left(1+\sqrt{2}\right)^2\right] \\ &= \frac{1}{4}\left[3 - 2\sqrt{6} + 2 + 1 + 2\sqrt{2} + 2\right] \\ &= \frac{1}{4}\left[8 - 2\sqrt{6} + 2\sqrt{2}\right] \\ &= \frac{1}{2}\left[4 - \sqrt{6} + \sqrt{2}\right] \end{aligned}$$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct proof	2
• Provides why – $\angle AOB = \frac{7\pi}{12}$ OR – $\angle AOC = \text{Arg}(w) - \text{Arg}(z+w)$ OR – $OACB$ is a rhombus OR – OC bisects $\angle AOB$ Or equivalent merit	1

Sample answer:



$$\angle AOB = \pi - \frac{\pi}{4} - \frac{\pi}{6} = \frac{12\pi - 3\pi - 2\pi}{12} = \frac{7\pi}{12}$$

$OACB$ is a parallelogram by definition of vector addition.

Since it has two adjacent sides of equal length, it is a rhombus.

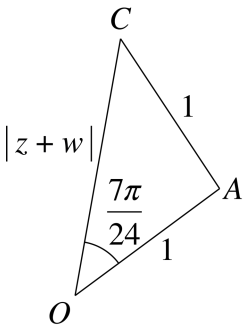
A diagonal of a rhombus bisects the angle at each vertex
so $\angle AOC = \frac{1}{2}\angle AOB = \frac{1}{2} \times \frac{7\pi}{12} = \frac{7\pi}{24}$

$$\angle AOC = \frac{7\pi}{24}$$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct proof	1

Sample answer:



$$\cos\frac{7\pi}{24} = \frac{1^2 + |z+w|^2 - 1^2}{2 \times 1 \times |z+w|}$$

$$= \frac{1}{2}|z+w|$$

$$= \frac{1}{2} \times \frac{\sqrt{4-\sqrt{6}+\sqrt{2}} \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}}$$

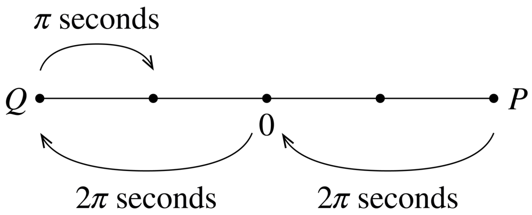
$$= \frac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{2 \times 2}$$

$$= \frac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{4}$$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Finds when the two particles first collide, or equivalent merit	2
• Obtains equation of motion for the first particle or sketches a relevant diagram related to the motion of the first particle or equivalent merit	1

Sample answer:



Since period is 8π, first particle is at 0 when second particle leaves P.

First particle is at Q when second particle is at 0.

$\therefore$ Time of collision is 5π after first particle is released.

Amplitude = 4 m, starts at maximum point

Period = 8π $\quad \therefore \quad \frac{2\pi}{4} = 8\pi$

$$n = \frac{1}{4}$$

$\therefore$ Equation of motion is $x = 4\cos\left(\frac{1}{4}t\right)$

When $t = 5\pi$, $x = 4\cos\left(\frac{5\pi}{4}\right)$

$$= -\frac{4}{\sqrt{2}}$$

$$= -2\sqrt{2}$$

$\therefore$ Collide $2\sqrt{2}$ metres left of 0 when $t = 5\pi$.

Alternative solution

Period $= 8\pi \quad \Rightarrow \quad$ Period $= \dfrac{2\pi}{n}$

$$n = \frac{2\pi}{8\pi} = \frac{1}{4}$$

Particle 1

$$X_1 = 4\cos\left(\frac{1}{4}t\right)$$

Particle 2

$$X_2 = 4\cos\left(\frac{1}{4}(t - 2\pi)\right) \qquad t \geq 2\pi$$

Particles meet if $X_1 = X_2$

$$\therefore \quad 4\cos\left(\frac{1}{4}t\right) = 4\cos\left(\frac{1}{4}(t - 2\pi)\right)$$

$$\therefore \quad \frac{1}{4}t = \frac{1}{4}(t - 2\pi) + 2k\pi$$

$$t = t - 2\pi + 8k\pi$$

$$2\pi = 8k\pi$$

Which is impossible since k is an integer.

OR

$$\frac{1}{4}t = -\frac{1}{4}(t - 2\pi) + 2k\pi$$

$$t = -t + 2\pi + 8k\pi$$

$$2t = 2\pi + 8k\pi$$

$$t = \pi + 4k\pi$$

- When $k = 0$, $t = \pi$ so the second particle has not been released yet.
- When $k = 1$, $t = 5\pi$ and that is the first time the particles collide.

$$X = 4\cos\left(\frac{5\pi}{4}\right) = 4 \times -\frac{1}{\sqrt{2}} = -2\sqrt{2}$$

$\therefore$ Meet when $t = 5\pi$ at $X = -2\sqrt{2}$

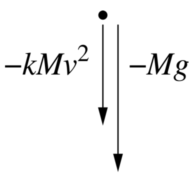
Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Obtains *x* as a function of *v* for upwards motion, or equivalent merit	2
• Obtains force equation for upwards part of flight, or equivalent merit	1

Sample answer:

Upward motion

$$M\ddot{x} = -kMv^2 - Mg \qquad t = 0,\ x = 0,\ \dot{x} = v_0$$



$$\ddot{x} = -kv^2 - g$$

$$v\frac{dv}{dx} = -kv^2 - g$$

$$\int -\frac{v\,dv}{kv^2 + g} = \int dx$$

$$-\frac{1}{2k}\ln\left|kv_0{}^2 + g\right| = x + c$$

At $x = 0$, $v = v_0$ therefore

$$-\frac{1}{2k}\ln\left|kv_0{}^2 + g\right| = x + c$$

$$\therefore \quad x = \frac{1}{2k}\ln\left|kv_0{}^2 + g\right| - \frac{1}{2k}\ln\left|kv^2 + g\right|$$

$$= \frac{1}{2k}\ln\left|\frac{kv_0{}^2 + g}{kv^2 + g}\right|$$

When $x = H$, $v = 0$

$$\therefore \quad H = \frac{1}{2k}\ln\left|\frac{kv_0{}^2 + g}{g}\right|$$

But $kv_0{}^2 + g$ and $g > 0$ so

$$H = \frac{1}{2k}\ln\left(\frac{kv_0{}^2 + g}{g}\right) \quad \text{——————} \ (1)$$

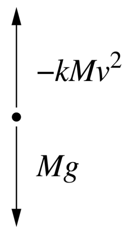
Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Equates correct expressions for *H* from parts (i) and (ii), with absolute values correctly dealt with, or equivalent merit	2
• Finds the force or acceleration equation of the downwards motion	1

Sample answer:

Downward motion (downward direction positive)

$x = 0,\ v = 0$



$$\therefore \quad M\ddot{x} = Mg - kMv^2$$

$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\int \frac{v\,dv}{g - kv^2} = \int dx$$

$$-\frac{1}{2k}\ln\left| g - kv^2 \right| = x + c$$

$$-\frac{1}{2k}\ln\left| g \right| = 0 + c \qquad x = 0,\ v = 0$$

$$\therefore \quad \frac{1}{2k}\ln\left| g \right| - \frac{1}{2k}\ln\left| g - kv^2 \right| = x$$

$$\frac{1}{2k}\ln\left| \frac{g}{g - kv^2} \right| = x$$

$x = H, \quad v = v_1$

$$\frac{1}{2k}\ln\left| \frac{g}{g - kv_1^{\,2}} \right| = H \qquad \text{———— (1)}$$

The particle never reaches terminal velocity so $g - kv_1^{\,2} > 0$

$$\frac{1}{2k}\ln\left(\frac{g}{g - kv_1^{\,2}} \right) = H \qquad \text{———— (2)}$$

Question 14 (c) (ii) (continued)

Equate equation (1) with (2)

$$\frac{1}{2k}\ln\left(\frac{kv_0^{\ 2}+g}{g}\right) = \frac{1}{2k}\ln\left(\frac{g}{g-kv_1^{\ 2}}\right)$$

$$\therefore \quad \frac{kv_0^{\ 2}+g}{g} = \frac{g}{g-kv_1^{\ 2}}$$

$$\left(kv_0^{\ 2}+g\right)\left(g-kv_1^{\ 2}\right) = g^2$$

$$kv_0^{\ 2}g - k^2v_0^{\ 2}v_1^{\ 2} + g^2 - gkv_1^{\ 2} = g^2$$

$$kv_0^{\ 2}g - k^2v_0^{\ 2}v_1^{\ 2} - gkv_1^{\ 2} = 0$$

$$kg\left(v_0^{\ 2}-v_1^{\ 2}\right) = k^2v_0^{\ 2}v_1^{\ 2}$$

$$g\left(v_0^{\ 2}-v_1^{\ 2}\right) = kv_0^{\ 2}v_1^{\ 2}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct proof	3
• Applies integration by parts using $\sin\theta \times \sin^{n-1}\theta$ and the limits of integration, or equivalent merit	2
• Writes $\sin^n\theta = \sin\theta \times \sin^{n-1}\theta$ and attempts integration by parts on these factors OR • Applies integration by parts, or equivalent merit	1

Sample answer:

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \sin^{n-1}\theta \, d\theta$$

$$= \left[-\cos\theta \sin^{n-1}\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos\theta)(n-1)\sin^{n-2}\theta \cos\theta \, d\theta$$

$$= 0 - 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta \, d\theta$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)\sin^{n-2}\theta \, d\theta$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - (n-1)\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$$

$$J_n = (n-1)J_{n-2} - (n-1)J_n$$

$$nJ_n = (n-1)J_{n-2}$$

$$J_n = \frac{n-1}{n}J_{n-2}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct proof	4
• Uses an appropriate substitution to obtain a correct definite integral from 0 to π in terms of $\phi = 2\theta$, or equivalent merit	3
• Obtains a definite integral in terms of $\sin 2\theta$, or equivalent merit	2
• Correctly uses given substitution to obtain a definite integral in terms of θ or equivalent merit	1

Sample answer:

$$I_n = \int_0^1 x^n (1-x)^n \, dx$$

If $x = \sin^2\theta$

then $\dfrac{dx}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta$

at $\quad x = 0, \quad \theta = 0$

at $\quad x = 1, \quad \theta = \dfrac{\pi}{2}$

$$\begin{aligned} I_n &= \int_0^1 x^n (1-x)^n \, dx \\ &= \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\left(1-\sin^2\theta\right)^n \sin 2\theta \, d\theta \\ &= \int_0^{\frac{\pi}{2}} \sin^{2n}\theta \cos^{2n}\theta \sin 2\theta \, d\theta \\ &= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n}(2\theta) \sin(2\theta) \, d\theta \\ &= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta) \, d\theta \end{aligned}$$

Question 15 (a) (ii) (continued)

If $\phi = 2\theta$

$$\frac{d\phi}{d\theta} = 2$$

at $\theta = 0$, $\phi = 0$

at $\theta = \frac{\pi}{2}$, $\phi = \pi$

$$I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\, d\theta$$

$$= \frac{1}{2^{2n}} \int_0^{\pi} \sin^{2n+1}\phi \times \frac{1}{2} d\phi$$

$$= \frac{1}{2^{2n+1}} \int_0^{\pi} \sin^{2n+1}\phi\, d\phi$$

but $\displaystyle\int_0^{\pi} \sin^{2n+1}\phi\, d\phi = 2\int_0^{\frac{\pi}{2}} \sin^{2n+1}\phi\, d\phi$

as $\sin^{2n+1}\phi$ is symmetrical about $\phi = \frac{\pi}{2}$

So $\displaystyle I_n = \frac{1}{2^{2n+1}} \int_0^{\pi} \sin^{2n+1}\phi\, d\phi$

$$= \frac{2}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\phi\, d\phi$$

$$= \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\phi\, d\phi$$

which can be rewritten

$$I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta\, d\theta$$

Question 15 (a) (iii)

Criteria	Marks
• Provides correct proof	2
• Obtains I_n in terms of J_{2n-1}, or equivalent merit	1

Sample answer:

$$I_n = \frac{1}{2^{2n}} \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta$$

$$= \frac{1}{2^{2n}} J_{2n+1}$$

$$= \frac{1}{2^{2n}} \left(\frac{(2n+1)-1}{2n+1} \right) J_{2n-1}$$

$$= \frac{1}{2^{2n}} \times \frac{2n}{2n+1} J_{2(n-1)+1}$$

But
$$\begin{aligned} I_n &= \frac{1}{2^{2n}} J_{2n+1} \\ 2^{2n} I_n &= J_{2n+1} \\ 2^{2(n-1)} I_{n-1} &= J_{2(n-1)+1} \\ &= J_{2n-1} \end{aligned}$$

$$I_n = \frac{1}{2^{2n}} \times \frac{2n}{2n+1} \times 2^{2(n-1)} I_{n-1}$$

$$= \frac{1}{2^2} \times \frac{2n}{2n+1} \times I_{n-1}$$

$$= \frac{n}{4n+2} I_{n-1}$$

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\overline{LP} = \overline{LB} + \overline{BC} + \overline{CP}$$

$$= \frac{1}{2}\underset{\sim}{b} + \left(\underset{\sim}{c} - \underset{\sim}{b}\right) + \frac{1}{2}\left(\underset{\sim}{d} - \underset{\sim}{c}\right)$$

$$= \frac{1}{2}\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{d} - \frac{1}{2}\underset{\sim}{c}$$

$$= -\frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{d}$$

$$= \frac{1}{2}\left(-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}\right)$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly expands LHS or RHS in terms of $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$ using dot products, or equivalent merit	2
• Obtains $\left\lvert \overrightarrow{LP} \right\rvert^2 = \lvert \underset{\sim}{c} \rvert^2 + 2\underset{\sim}{c} \cdot \underset{\sim}{d} - 2\underset{\sim}{c} \cdot \underset{\sim}{b} + \lvert \underset{\sim}{d} - \underset{\sim}{b} \rvert^2$, or equivalent merit	1

Sample answer:

$$\overrightarrow{MQ} = \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c})$$

$$\overrightarrow{NR} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{d})$$

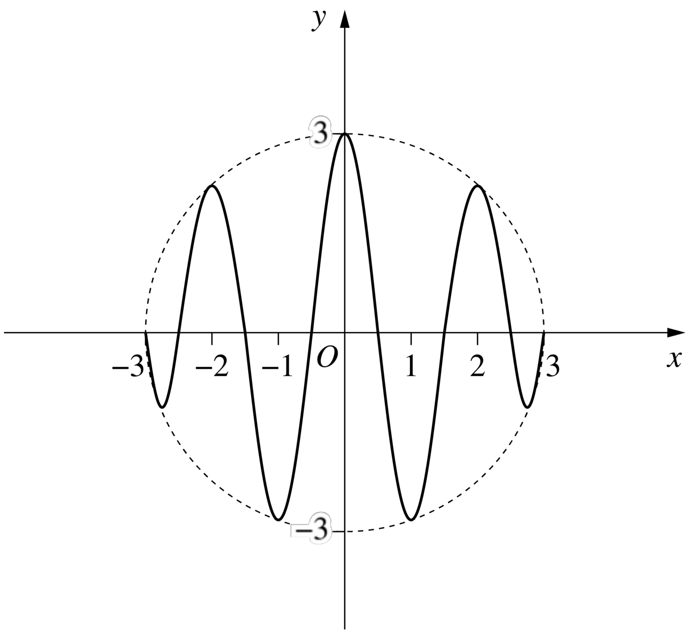
$$\begin{aligned}
\text{RHS} &= 4\left(\left\lvert \overrightarrow{LP} \right\rvert^2 + \left\lvert \overrightarrow{MQ} \right\rvert^2 + \left\lvert \overrightarrow{NR} \right\rvert^2 \right) \\
&= 4\left(\overrightarrow{LP} \cdot \overrightarrow{LP} + \overrightarrow{MQ} \cdot \overrightarrow{MQ} + \overrightarrow{NR} \cdot \overrightarrow{NR} \right) \\
&= 4\left(\frac{1}{4}(-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}) \cdot (-\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}) + \right. \\
&\qquad \frac{1}{4}(\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c}) \cdot (\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c}) + \\
&\qquad \left. \frac{1}{4}(\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{d}) \cdot (\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{d}) \right) \\
&= \left(\underset{\sim}{c} + (\underset{\sim}{d} - \underset{\sim}{b})\right) \cdot \left(\underset{\sim}{c} + (\underset{\sim}{d} - \underset{\sim}{b})\right) + \left(\underset{\sim}{d} + (\underset{\sim}{b} - \underset{\sim}{c})\right) \cdot \left(\underset{\sim}{d} + (\underset{\sim}{b} - \underset{\sim}{c})\right) + \left(\underset{\sim}{b} + (\underset{\sim}{c} - \underset{\sim}{d})\right) \cdot \left(\underset{\sim}{b} + (\underset{\sim}{c} - \underset{\sim}{d})\right) \\
&= \lvert \underset{\sim}{c} \rvert^2 + 2\underset{\sim}{c} \cdot (\underset{\sim}{d} - \underset{\sim}{b}) + \lvert \underset{\sim}{d} - \underset{\sim}{b} \rvert^2 + \\
&\qquad \lvert \underset{\sim}{d} \rvert^2 + 2\underset{\sim}{d} \cdot (\underset{\sim}{b} - \underset{\sim}{c}) + \lvert \underset{\sim}{b} - \underset{\sim}{c} \rvert^2 + \\
&\qquad \lvert \underset{\sim}{b} \rvert^2 + 2\underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{d}) + \lvert \underset{\sim}{c} - \underset{\sim}{d} \rvert^2 \\
&= \lvert \underset{\sim}{b} \rvert^2 + \lvert \underset{\sim}{c} \rvert^2 + \lvert \underset{\sim}{d} \rvert^2 + \lvert \underset{\sim}{b} - \underset{\sim}{c} \rvert^2 + \lvert \underset{\sim}{d} - \underset{\sim}{b} \rvert^2 + \lvert \underset{\sim}{c} - \underset{\sim}{d} \rvert^2 + \\
&\qquad 2\left[\underset{\sim}{c} \cdot \underset{\sim}{d} - \underset{\sim}{c} \cdot \underset{\sim}{b} + \underset{\sim}{d} \cdot \underset{\sim}{b} - \underset{\sim}{d} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{d}\right] \\
&= \left\lvert \overrightarrow{AB} \right\rvert^2 + \left\lvert \overrightarrow{AC} \right\rvert^2 + \left\lvert \overrightarrow{AD} \right\rvert^2 + \left\lvert \overrightarrow{BC} \right\rvert^2 + \left\lvert \overrightarrow{BD} \right\rvert^2 + \left\lvert \overrightarrow{CD} \right\rvert^2 + 2 \times 0 \\
&= \text{LHS}
\end{aligned}$$

Question 15 (c)

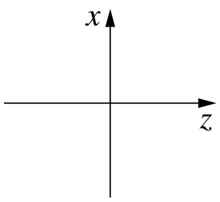
Criteria	Marks
• Provides correct solution	3
• Provides parametric equations of a curve that lies on the sphere, or equivalent merit	2
• Sketches the graph $y = f(x)g(x)$, or equivalent merit	1

Sample answer:

Graph of $y = \cos(\pi x)\sqrt{9 - x^2}$ for $-3 \le x \le 3$



Projecting curve $\mathscr{C}$ on sphere onto xz-plane gives the curve above



So if $z = t$

$$x = \cos(\pi t)\sqrt{9 - t^2}$$

Similarly, projecting curve onto yz-plane gives $y = -\sin(\pi t)\sqrt{9 - t^2}$

$\therefore$ Parametric equations are

$$x = \cos(\pi t)\sqrt{9 - t^2}$$
$$y = -\sin(\pi t)\sqrt{9 - t^2}$$
$$z = t$$

Question 15 (c) (continued)

Alternative solution

Because the curve lies on the sphere, we must have $x^2 + y^2 + z^2 = 3^2$

Given the hint about using $\sqrt{9-t^2}\cos(\pi t)$ and noticing that

$$\left(\sqrt{9-t^2}\cos(\pi t)\right)^2 + \left(\sqrt{9-t^2}\sin(\pi t)\right)^2 + t^2 = 3^2$$

as well as the fact that z increases as the point travels on the curve whereas x and y change signs, one can try the following

$z = t$

One of x and y is $\pm\sqrt{9-t^2}\cos(\pi t)$ and the other one is $\pm\sqrt{9-t^2}\sin(\pi t)$.

- We notice that when $t = z = 0$, x is positive and reaches a maximum value so $x(t) = \sqrt{9-t^2}\cos(\pi t)$.
- This leaves $y = \sqrt{9-t^2}\sin(\pi t)$ or $y = -\sqrt{9-t^2}\sin(\pi t)$. Given that shortly after when $t = z = 0$, $y < 0$ we get $y(t) = -\sqrt{9-t^2}\sin(\pi t)$.

Finally $x = \sqrt{9-t^2}\cos(\pi t)$

$y = -\sqrt{9-t^2}\sin(\pi t)$

$z = t$

Question 16 (a) (i)

Criteria	Marks
• Provides correct proof	2
• Recognises that $1+w+w^2$ is a factor of $1-w^3$, or equivalent merit	1

Sample answer:

As $w \neq 1$

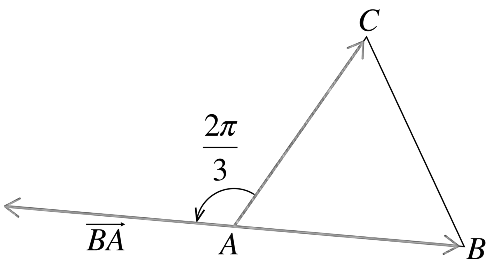
$$1+w+w^2 = \frac{1-w^3}{1-w} = 0 \quad \text{since} \quad w^3 = \left(e^{\frac{2i\pi}{3}}\right)^3 = 1$$

Question 16 (a) (ii)

Criteria	Marks
• Attempts to use the fact that $\overline{AC}$ rotated by $\frac{2\pi}{3}$ is $\overline{BA}$, or equivalent merit	1
• Provides correct proof	2

Sample answer:

Suppose triangle ABC is equilateral and anticlockwise



Rotating $\overline{AC}$ by $+\frac{2\pi}{3}$ brings it onto $\overline{BA}$ so $w(c-a) = a-b$

$$a(1+w) - b - wc = 0$$

$$\Rightarrow \quad -aw^2 - b - wc = 0 \qquad \text{(by part (i))}$$

$$\Rightarrow \quad a + bw + w^2 c = 0 \qquad \text{(multiplying by } -w\text{)}$$

Question 16 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Identifies that in either situation the product is 0, or equivalent merit	1

Sample answer:

Suppose ABC is equilateral.

It is either anticlockwise or clockwise.

Therefore $a + bw + cw^2 = 0$ or $a + bw^2 + cw = 0$

This is equivalent to $\left(a + bw + cw^2\right)\left(a + bw^2 + cw\right) = 0$

Since a product is zero if and only if one of its factors is zero,

$$\begin{aligned}
& \left(a + bw + cw^2\right)\left(a + bw^2 + cw\right) = 0 \\
\Leftrightarrow \quad & a^2 + bw^3 + cw^3 + ab\left(w^2 + w\right) + bc\left(w^2 + w^4\right) + ac\left(w + w^2\right) = 0 \\
\Leftrightarrow \quad & a^2 + b^2 + c^2 - (ab + bc + ac) = 0 \qquad \text{(since } w^3 = 1 \text{ and } w + w^2 = -1 \text{ by part (i))} \\
\Leftrightarrow \quad & a^2 + b^2 + c^2 = ab + bc + ac \qquad \text{as required}
\end{aligned}$$

Question 16 (b) (i)

Criteria	Marks
• Provides correct proof	2
• Defines a function $f(x) = x - \ln x$, or equivalent merit	1

Sample answer:

Let $f(x) = x - \ln x$ for $x > 0$

$f'(x)$ has the same sign as $x - 1$ when $x > 0$

t	0	1	$-\infty$
$f'(t)$	$-$	0	$+$
f	$\searrow$	1	$\nearrow$

This shows that $\forall x > 0 \quad f(x) \geq 1$
that is, $x - \ln x \geq 1$
This implies $\forall x > 0 \quad x - \ln x > 0$ so $x > \ln x$

<u>Alternative explanation</u>

$$f'(x) = 1 - \frac{1}{x}$$

$$f''(x) = \frac{1}{x^2} > 0, \qquad \forall x > 0$$

$\therefore$ Curve is concave up for all $x > 0$

$$f'(x) \geq 0 \quad \text{when} \quad 1 \geq \frac{1}{x}$$

$$x \geq 1 \quad (\text{since } x > 0)$$

$$\begin{aligned} f(1) &= 1 - \ln 1 \\ &= 1 \end{aligned}$$

$$\therefore \quad f(x) \geq 1 \qquad \forall x > 0$$
$$> 0$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct proof	3
• Applies part (i) to each term of $\sum_{i=1}^{n} \ln i$ OR • Establishes inductive step OR • Recognises that the question is equivalent to showing $n^2 + n > 2\left(\sum_{i=1}^{n} \ln i\right)$	2
• Uses part (i) to establish base case of induction, or equivalent merit	1

Sample answer:

Assume n is a positive integer

$$\left.\begin{aligned} n &> \ln n \\ n-1 &> \ln(n-1) \\ &\vdots \\ 2 &> \ln(2) \\ 1 &> \ln(1) \end{aligned}\right\} \text{by part (i)}$$

Adding all those inequalities gives

$$1 + 2 + \cdots + (n-1) + n > \ln(n) + \ln(n-1) + \cdots + \ln(2) + \ln 1$$

$$\frac{n(n+1)}{2} > \ln(n!)$$

$$n^2 + n > 2\ln(n!) = \ln\left((n!)^2\right)$$

By taking the exponential of both sides, we get

$$e^{n^2+n} > (n!)^2$$

as required.

Question 16 (c)

Criteria	Marks
• Provides correct sketch	3
• Identifies the quadrant in which the region lies, or equivalent merit	2
• Identifies a quadrant that is excluded, or equivalent merit	1

Sample answer:

Let $\theta = \operatorname{Arg}\left(\frac{z}{w}\right)$, then $\frac{\pi}{2} < \theta < \pi$

As $\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = 1$ $\qquad \frac{w}{z} = \overline{\left(\frac{z}{w}\right)}$

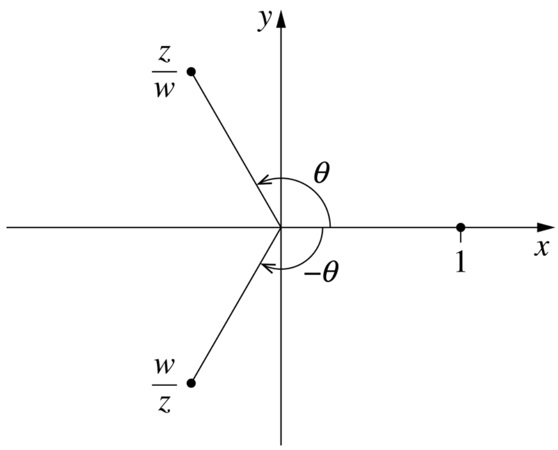
So $\operatorname{Arg}\left(\frac{w}{z}\right) = -\theta$

$$\frac{xz + yw}{z} = x + y\frac{w}{z}$$

$$= x\,1 + y\frac{w}{z}$$

For $\frac{\pi}{2} < \operatorname{Arg}\left(\frac{xz + yw}{z}\right) < \pi$

the number $x\,1 + y\frac{w}{z}$ must lie in the second quadrant of the complex plane.



Clearly $y < 0$, otherwise $x + y\frac{w}{z}$ lies below the x-axis.

The real part of $x + y\frac{w}{z}$ must be negative.

Question 16 (c) (continued)

We have $\operatorname{Re}\left(\frac{w}{z}\right) = \cos(-\theta)$

So $\quad x + y\cos(-\theta) < 0$

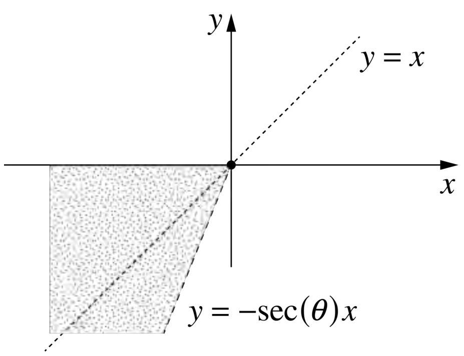
$\quad x < -\cos(\theta)\, y \qquad$ cos is even

$\quad -1 < \cos(\theta) < 0 \qquad \left(\frac{\pi}{2} < \theta < \pi\right)$

So $\quad 0 < -\cos(\theta) < 1$

$\Rightarrow \quad -\sec(\theta) > 1$

and $y > -\sec(\theta)\, x$



2023 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-N1 Introduction to Complex Numbers	MEX12-7
2	1	MEX-P1 The Nature of Proof	MEX12-2
3	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
4	1	MEX-P1 The Nature of Proof	MEX12-8
5	1	MEX-V1 Further Work With Vectors	MEX12-3
6	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
7	1	MEX-N1 Introduction to Complex Numbers	MEX12-4
8	1	MEX-N2 Using Complex Numbers	MEX12-4
9	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
10	1	MEX-V1 Further Work With Vectors	MEX12-8

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	MEX-N1 Introduction to Complex Numbers	MEX12-7
11 (b)	3	MEX-V1 Further Work with Vectors	MEX12-3
11 (c)	2	MEX-V1 Further Work with Vectors	MEX12-3
11 (d)	2	MEX-V1 Further Work with Vectors	MEX12-7
11 (e)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
11 (f)	4	MEX-C1 Further Integration	MEX12-5
12 (a)	3	MEX-P1 The Nature of Proof	MEX12-2
12 (b)	2	MEX-P1 The Nature of Proof	MEX12-2
12 (c) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
12 (c) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
12 (d)	3	MEX-N2 Using Complex Numbers	MEX12-1
12 (e) (i)	1	MEX-N2 Using Complex Numbers	MEX12-7
12 (e) (ii)	2	MEX-N2 Using Complex Numbers	MEX12-7
13 (a)	3	MEX-C1 Further Integration	MEX12-5
13 (b) (i)	1	MEX-P1 The Nature of Proof	MEX12-2
13 (b) (ii)	3	MEX-P2 Further Proof by Mathematical Induction	MEX12-2
13 (c) (i)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
13 (c) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
13 (c) (iii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
13 (c) (iv)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7

Question	Marks	Content	Syllabus outcomes
14 (a) (i)	3	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4
14 (a) (ii)	2	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4
14 (a) (iii)	1	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4
14 (b)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
14 (c) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
14 (c) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (a) (i)	3	MEX-C1 Further Integration	MEX12-5
15 (a) (ii)	4	MEX-C1 Further Integration	MEX12-5
15 (a) (iii)	2	MEX-C1 Further Integration	MEX12-5
15 (b) (i)	1	MEX-V1 Further Work with Vectors	MEX12-3
15 (b) (ii)	3	MEX-V1 Further Work with Vectors	MEX12-3
15 (c)	3	MEX-V1 Further Work with Vectors	MEX12-3
16 (a) (i)	2	MEX-N2 Using Complex Numbers	MEX12-4
16 (a) (ii)	2	MEX-V1 Further Work with Vectors MEX-N2 Using Complex Numbers	MEX12-3, MEX12-4
16 (a) (iii)	2	MEX-N2 Using Complex Numbers	MEX12-4
16 (b) (i)	2	MEX-P1 The Nature of Proof	MEX12-1
16 (b) (ii)	3	MEX-P1 The Nature of Proof	MEX12-2
16 (c)	3	MEX-N2 Using Complex Numbers	MEX12-4