2022 HSC Mathematics Extension 2
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	D
3	B
4	C
5	C
6	A
7	D
8	B
9	D
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Multiplies top and bottom by the conjugate of the denominator, or equivalent merit	1

Sample answer:

$$\frac{3-i}{2+i} = \frac{3-i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{6-2i-3i-1}{4+1}$$

$$= \frac{5-5i}{5}$$

$$= 1-i$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Chooses a suitable substitution, or equivalent merit	1

Sample answer:

$$\int \sin^3 2x \cos 2x \, dx$$

$$= \frac{1}{4} \int 4 \sin^3 2x \cos 2x \, dx$$

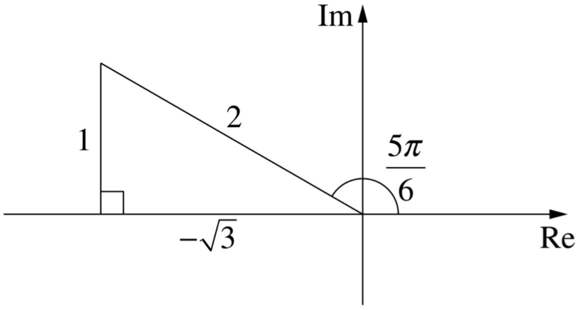
$$= \frac{1}{4}\left(\frac{\sin^4(2x)}{2}\right) + C$$

$$= \frac{1}{8}\sin^4(2x) + C$$

Question 11 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains correct modulus OR • Obtains correct argument	1

Sample answer:



$$-\sqrt{3}+i=2e^{i\frac{5\pi}{6}}$$

Question 11 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Attempts to apply de Moivre's theorem, or equivalent merit	1

Sample answer:

$$\begin{aligned}\left(-\sqrt{3}+i\right)^{10} &= \left(2e^{i\frac{5\pi}{6}}\right)^{10} \qquad \text{from part (i)}\\ &= 2^{10}e^{i\frac{50\pi}{6}}\\ &= 2^{10}e^{i\frac{\pi}{3}}\\ &= 2^{10}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= 2^9+2^9\sqrt{3}i\end{aligned}$$

Question 11 (d)

Criteria	Marks
• Provides correct solution	3
• Obtains correct dot product OR attempts to use the cosine rule using 3 correct lengths	2
• Obtains one relevant vector OR obtains one relevant length	1

Sample answer:

$A(1, -1, 2) \qquad B(0, 2, -1) \qquad C(2, 1, 1)$

$$\underset{\sim}{a} = \overrightarrow{BA} = \begin{pmatrix} 1 - 0 \\ -1 - 2 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{b} = \overrightarrow{BC} = \begin{pmatrix} 2 - 0 \\ 1 - 2 \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \cos\angle ABC &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{b}|} = \frac{2+3+6}{\sqrt{1+9+9} \cdot \sqrt{4+1+4}} \\ &= \frac{11}{3\sqrt{19}} \end{aligned}$$

$$\begin{aligned} \therefore\ \angle ABC &= \cos^{-1}\left(\frac{11}{3\sqrt{19}}\right) \\ &\approx 33^\circ \end{aligned}$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Finds equation of ℓ_2 in any form OR finds the correct slope of ℓ_2 OR equivalent merit	1

Sample answer:

ℓ_1: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\ell_2 \parallel \ell_1$ passes through (–6, 5)

ℓ_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$x = -6 + 3\mu$$
$$y = 5 + 2\mu$$

$$2x = -12 + 6\mu$$
$$3y = 15 + 6\mu$$

Subtract

$$3y - 2x = 27$$
$$3y = 2x + 27$$
$$y = \frac{2}{3}x + 9$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Obtains integrand in terms of t in simplest form, or equivalent merit	2
• Correctly replaces dx in terms of t and dt, or equivalent merit	1

Sample answer:

$$t = \tan\frac{x}{2} \qquad \Rightarrow \qquad dx = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{1+\cos x - \sin x}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}$$

$$= \int \frac{2dt}{2-2t}$$

$$= \int \frac{dt}{1-t}$$

$$= -\ln\left|1-t\right| + k$$

$$= -\ln\left|1-\tan\frac{x}{2}\right| + k$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	2
• Chooses a suitable strategy	1

Sample answer:

$a,\ b \geq 0$ so $\sqrt{a}$, $\sqrt{b}$ are real

$$\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$$

$$a - 2\sqrt{a}\sqrt{b} + b \geq 0$$

$$a + b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Question 12 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains the displacement of the particle at time t, or equivalent merit	2
• Obtains the position of the particle at time t OR • Finds the time at which the velocity of the particle is a maximum	1

Sample answer:

$\ddot{x} = 12 - 6t$

$t = 0, \quad v = 0, \quad x = 0$

$$\frac{dv}{dt} = 12 - 6t$$

$$v = \int 12 - 6t\,dt$$

$$= 12t - 3t^2 + c$$

$t = 0, \quad v = 0, \quad \Rightarrow \quad c = 0$

$\therefore \quad v = 12t - 3t^2$

$$\frac{dx}{dt} = 12t - 3t^2$$

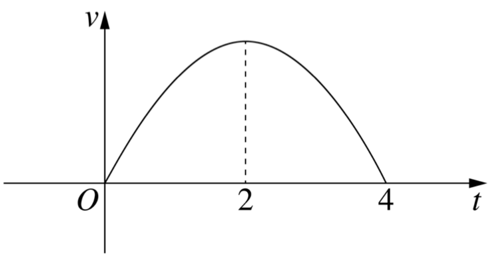
$\therefore \quad x = 6t^2 - t^3 + c$

$t = 0, \quad v = 0, \quad \Rightarrow \quad c = 0$

$\therefore \quad x = 6t^2 - t^3$

<u>Maximum velocity</u>

$v = 12t - 3t^2$



v is maximum when $t = 2$.

$\therefore$ Position at maximum velocity is

$x = 6 \times 2^2 - 2^3$

$= 16$ units to right of origin

Question 12 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$F = ma = 1 \times a = a$$

$$a = -\left(v + 3v^2\right)$$

$$v\frac{dv}{dx} = -v(1+3v) \qquad \text{and } v \neq 0$$

$$\frac{dv}{dx} = -(1+3v)$$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Separates the variables from the DE in part (i), or equivalent merit	1

Sample answer:

$$\frac{dv}{dx} = -(1+3v)$$

$$\int \frac{dv}{1+3v} = -\int dx$$

$$\frac{1}{3}\ln|1+3v| = -x + c$$

When $x = 0$, $v = u$

So $\frac{1}{3}\ln|1+u| = c$

$$\therefore \quad x = \frac{1}{3}\ln|1+3u| - \frac{1}{3}\ln|1+3v|$$

$$= \frac{1}{3}\ln\left(\frac{1+3u}{1+3v}\right)$$

Since $1 + 3v > 0$ and $1 + 3u > 0$

Question 12 (d)

Criteria	Marks
• Provides correct solution	4
• Obtains correct primitive	3
• Obtains the correct partial fraction decomposition, or equivalent merit	2
• Writes the correct general form of the partial fraction decomposition, or equivalent merit	1

Sample answer:

$$\frac{4+x}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$$

$$4+x = A(4+x^2) + (Bx+C)(1-x)$$

$x=1 \qquad 5 = 5A, \qquad A = 1$

$$\frac{4+x}{(1-x)(4+x^2)} - \frac{1}{1-x} = \frac{4+x-(4+x^2)}{(1-x)(4+x^2)}$$

$$= \frac{x-x^2}{(1-x)(4+x^2)}$$

$$= \frac{x}{4+x^2}$$

$$\int_2^n \frac{4+x}{(1-x)(4+x^2)}\,dx = \int_2^n \frac{1}{1-x} + \frac{x}{4+x^2}\,dx$$

$$= \int_2^n \frac{1}{1-x} + \frac{1}{2}\left(\frac{2x}{4+x^2}\right)dx$$

$$= \left[-\ln|1-x| + \frac{1}{2}\ln|4+x^2|\right]_2^n$$

$$= -\ln|1-n| + \frac{1}{2}\ln|4+n^2| - \frac{1}{2}\ln 8$$

$$= \ln\frac{1}{|n-1|} + \frac{1}{2}\ln|4+n^2| - \frac{1}{2}\ln 8$$

$$= \frac{1}{2}\left(2\ln\frac{1}{|n-1|} + \ln|4+n^2| - \ln 8\right)$$

$$= \frac{1}{2}\ln\left(\frac{4+n^2}{8|n-1|^2}\right)$$

Question 12 (e)

Criteria	Marks
• Provides correct solution	3
• Uses the properties of complex numbers with modulus 1 to simplify their correct fraction, or equivalent merit	2
• Divides the numerator and denominator by a suitable power of z, or equivalent merit	1

Sample answer:

If $z = e^{i\theta}$, then $\dfrac{1}{z} = e^{-i\theta}$

$$w = \frac{z^2 - 1}{z^2 + 1}$$

$$= \frac{z\left(z - \frac{1}{z}\right)}{z\left(z + \frac{1}{z}\right)}$$

$$= \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

$$= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}$$

$$= \frac{2i\sin\theta}{2\cos\theta}$$

$= i\tan\theta$ which is purely imaginary.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Uses the assumption that $n = 2k$ to factor $2^n - 1$, or equivalent merit	2
• Attempts to use the contrapositive, or equivalent merit	1

Sample answer:

It is equivalent to prove the contrapositive: 'if n is even, then $2^n - 1$ is not prime'.

Assume n is even. With $n \geq 3$, that means $n = 2k$ where k is an integer with $k \geq 2$.

$$2^n - 1 = 2^{2k} - 1 = \left(2^k\right)^2 - 1 = \left(2^k + 1\right)\left(2^k - 1\right)$$

With $k \geq 2$, $2^k + 1 \geq 5$ and $2^k - 1 \geq 3$, so $2^n - 1$ has two proper factors so it is not prime.

Question 13 (b)

Criteria	Marks
• Provides correct proof	4
• Proves the inductive step, or equivalent merit	3
• Establishes the base case and makes some progress with the inductive step, or equivalent merit	2
• Establishes base case, or equivalent merit	1

Sample answer:

$a_1 = \sqrt{2} \qquad a_2 = \sqrt{2+\sqrt{2}} \qquad a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \qquad \ldots$

$a_{n+1}^2 = 2 + a_n, \qquad n \geq 1$

Consider $n = 1$ case

RHS $= 2\cos\dfrac{\pi}{4} = 2 \times \dfrac{1}{\sqrt{2}} = \sqrt{2} = a_1 =$ LHS, so the result is true for $n = 1$

Assume the result is true for $n = k$.

Thus, $a_k = 2\cos\dfrac{\pi}{2^{k+1}}, \qquad k \geq 1$

Prove the result is true for $n = k + 1$

$$\begin{aligned} a_{k+1}^2 &= 2 + a_k \\ &= 2 + 2\cos\frac{\pi}{2^{k+1}} \\ &= 2\left[1 + \cos\frac{\pi}{2^{k+1}}\right] \qquad\qquad 1+\cos\theta = 2\cos^2\frac{\theta}{2} \\ &= 2 \times 2\cos^2\frac{\pi}{2^{k+2}} \\ &= 4\cos^2\frac{\pi}{2^{k+2}} \end{aligned}$$

$$\therefore\ a_{k+1} = 2\cos\frac{\pi}{2^{k+2}} \qquad\qquad \left(a_{k+1} > 0\right)$$

Thus if the result is true for k, it is also true for $k + 1$.

Hence, using the principle of mathematical induction, the result is true for all $k \geq 1$.

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Writes -1 in general exponential form, or equivalent merit	1

Sample answer:

$$z^5 + 1 = 0$$

$$z^5 = -1 = e^{i\pi}$$

$$\therefore \quad z = e^{i\left(\frac{2k\pi + \pi}{5}\right)}, \qquad k = 0, \quad \pm 1, \quad \pm 2$$

$$\therefore \quad z = e^{i\frac{\pi}{5}}, \quad z = e^{i\frac{3\pi}{5}}, \quad z = e^{-\frac{\pi}{5}i}, \quad z = e^{i\frac{\pi}{5}}, \quad z = e^{i\pi} = -1$$

$$z = -1, \quad e^{\pm i\frac{\pi}{5}}, \quad e^{\pm i\frac{3\pi}{5}}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Uses the given substitution and attempts to simplify, or equivalent merit	1

Sample answer:

Let z be a solution to $z^5 + 1 = 0$ with $z \neq -1$.

Let $u = z + \frac{1}{z}$

$$\begin{aligned}
u^2 - u - 1 &= \left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) - 1 \\
&= z^2 + 2 + \frac{1}{z^2} - z - \frac{1}{z} - 1 \\
&= z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} \\
&= \frac{z^4 - z^3 + z^2 - z + 1}{z^2} \\
&= \frac{1}{z^2}\left(\frac{1-(-z)^5}{1-(-z)}\right) && \text{using Geometric series, as } z \neq -1 \\
&= \frac{1+z^5}{z^2(1+z)} \\
&= 0 && \text{since } z^5 + 1 = 0
\end{aligned}$$

$\therefore\ u = z + \frac{1}{z}$ is a solution of $u^2 - u - 1 = 0$

Question 13 (c) (iii)

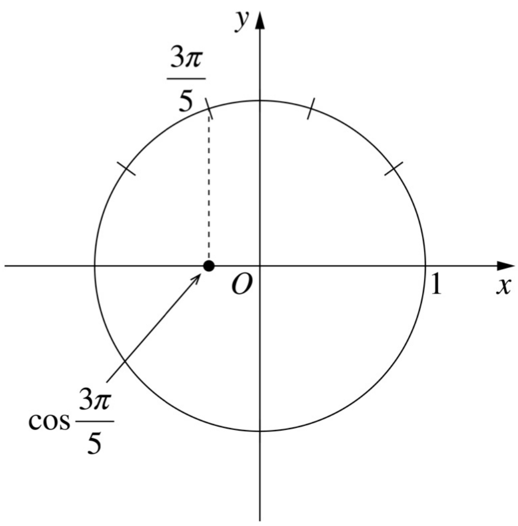
Criteria	Marks
• Provides correct solution	3
• Uses the connection between one of the solutions in part (i) and $\cos\frac{3\pi}{5}$	2
• Finds the values of u, or equivalent merit	1

Sample answer:

By part (i), $z = e^{\frac{3i\pi}{5}}$ is a solution of $z^5 + 1 = 0$.

Since $e^{\frac{3i\pi}{5}} \neq -1$, by part (ii), $u = e^{\frac{3i\pi}{5}} + \frac{1}{e^{\frac{3i\pi}{5}}} = 2\cos\frac{3\pi}{5}$ is a solution of $u^2 - u - 1 = 0$.

This equation has two solutions, $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.



$\cos\frac{3\pi}{5} < 0$ whereas $\frac{1+\sqrt{5}}{2} > 0$

So $\quad 2\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{2}$

$\therefore \quad \cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$

Question 14 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use a proof by contradiction or contrapositive, or equivalent merit	1

Sample answer:

Assume $\lambda\vec{u} + \mu\vec{v} = \vec{0}$

If λ and μ are both non-zero, then $\vec{u} = -\dfrac{\mu}{\lambda}\vec{v}$ and $\vec{v}$ are parallel.

But $\vec{u}$, $\vec{v}$ are not parallel.

Hence, one of λ or μ is zero.

Assume the other one is not zero.

Without loss of generality, assume $\lambda = 0$ but $\mu \neq 0$ then $\mu\vec{v} = \vec{0}$.

But $\vec{v} \neq \vec{0}$ so $\mu = 0$, a contradiction.

Hence $\lambda = \mu = 0$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

If $\lambda_1\vec{u} + \mu_1\vec{v} = \lambda_2\vec{u} + \mu_2\vec{v}$

Then $(\lambda_1 - \lambda_2)\vec{u} + (\mu_1 - \mu_2)\vec{v} = \vec{0}$

So $\lambda_1 - \lambda_2 = 0$ and $\mu_1 - \mu_2 = 0$ From part (i)

And hence $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Writes $\overrightarrow{SL}$ or $\overrightarrow{BL}$ as a linear combination of $\overrightarrow{SB}$ and $\overrightarrow{SC}$, or equivalent merit	1

Sample answer:

L is on the line SK so $\overrightarrow{SL} = \lambda\overrightarrow{SK}$ for some real number λ.

$$\overrightarrow{SL} = \frac{\lambda}{4}\overrightarrow{SB} + \frac{\lambda}{3}\overrightarrow{SC}$$

Similarly, there exists a real number μ such that

$$\overrightarrow{BL} = \mu\overrightarrow{BC} = -\mu\overrightarrow{SB} + \mu\overrightarrow{SC}$$

$$\begin{aligned}\overrightarrow{SL} &= \overrightarrow{SB} + \overrightarrow{BL} \\ &= (1-\mu)\overrightarrow{SB} + \mu\overrightarrow{SC}\end{aligned}$$

Now, $\dfrac{\lambda}{4}\overrightarrow{SB} + \dfrac{\lambda}{3}\overrightarrow{SC} = (1-\mu)\overrightarrow{SB} + \mu\overrightarrow{SC}$

$\overrightarrow{SB}$ and $\overrightarrow{SL}$ are not parallel, so from part (ii),

$$\frac{\lambda}{4} = 1-\mu \quad \text{and} \quad \frac{\lambda}{3} = \mu$$

$$\begin{aligned}\lambda &= 3\mu \\ \lambda &= 4-4\mu \\ 3\mu &= 4-4\mu \\ \mu &= \frac{4}{7}\end{aligned}$$

Hence $\overrightarrow{BL} = \mu\overrightarrow{BC} = \dfrac{4}{7}\overrightarrow{BC}$

Question 14 (a) (iv)

Criteria	Marks
• Provides correct solution	2
• Finds the vector $\overrightarrow{AL}$, or equivalent merit	1

Sample answer:

Write both $\overrightarrow{AP}$ and $\overrightarrow{AL}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\begin{aligned}\overrightarrow{AL} &= \overrightarrow{AB} + \overrightarrow{BL} \\ &= \overrightarrow{AB} + \frac{4}{7}\overrightarrow{BC} \\ &= \overrightarrow{AB} + \frac{4}{7}\left(\overrightarrow{BA} + \overrightarrow{AC}\right) \\ &= \frac{3}{7}\overrightarrow{AB} + \frac{4}{7}\overrightarrow{AC}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AP} &= -6\overrightarrow{AB} - 8\overrightarrow{AC} \\ &= -14\left(\frac{3}{7}\overrightarrow{AB} + \frac{4}{7}\overrightarrow{AC}\right) \\ &= -14\overrightarrow{AL}\end{aligned}$$

$\overrightarrow{AP} = -14\overrightarrow{AL}$, so $\overrightarrow{AP}$ and $\overrightarrow{AL}$ are parallel, and A is common to those vectors, so P lies on the line AL.

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$J_n = \int_0^1 x^n e^{-x}\, dx \qquad n \geq 0$$

$$n = 0 \qquad J_0 = \int_0^1 e^{-x}\, dx$$

$$= -\left[e^{-x}\right]_0^1$$

$$= -\left(\frac{1}{e} - 1\right)$$

$$= 1 - \frac{1}{e}$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• States that $e^{-x} \leq 1$ for $0 \leq x \leq 1$, or equivalent merit	1

Sample answer:

$$J_n = \int_0^1 x^n e^{-x}\, dx \leq \int_0^1 x^n\, dx \qquad \text{since } x^n e^{-x} \leq x^n \text{ for } [0,\, 1]$$

$$= \left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{n+1}$$

$$\therefore\ J_n \leq \frac{1}{n+1}$$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Attempts to apply integration by parts, correctly dealing with the limits of integration, or equivalent merit	1

Sample answer:

$$J_n = \int_0^1 x^n e^{-x}\, dx$$

$$= \left[x^n \left(-e^{-x} \right) \right]_0^1 - \int_0^1 n x^{n-1} \left(-e^{-x} \right) dx$$

$$= -\frac{1}{e} + n\, J_{n-1}$$

Question 14 (b) (iv)

Criteria	Marks
• Provides correct solution	2
• Uses part (iii) to write J_{k+1} in terms of J_k, or equivalent merit	1

Sample answer:

$$J_n = -\frac{1}{e} + n \times J_{n-1} \qquad n \geq 0$$

$n = 0$

$$\text{RHS} = 0! - \frac{0!}{e}\sum_{r=0}^{0}\frac{1}{1!}$$

$$= 1 - \frac{1}{e} = J_0 \qquad \text{by part (i)}$$

$$= \text{LHS}$$

$\therefore$ The property is true for $n = 0$.

Assume the property holds for k, so $J_k = k! - \dfrac{k!}{e}\displaystyle\sum_{r=0}^{k}\frac{1}{r!}$

By part (iii)

$$J_{k+1} = (k+1)J_k - \frac{1}{e}$$

$$= (k+1)\left(k! - \frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right) - \frac{1}{e}$$

$$= (k+1)! - \left(\frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right) - \frac{1}{e}\frac{(k+1)!}{(k+1)!}$$

$$= (k+1)! - \frac{(k+1)!}{e}\left(\left(\sum_{r=0}^{k}\frac{1}{r!}\right) + \frac{1}{(k+1)!}\right)$$

$$= (k+1)! - \frac{(k+1)!}{e}\left(\sum_{r=0}^{k+1}\frac{1}{r!}\right)$$

Thus, the result is true for $n = k + 1$, if true for $n = k$.

Thus, using the principle of mathematical induction, the result is proven.

Question 14 (b) (v)

Criteria	Marks
• Provides correct solution	1

Sample answer:

From part (ii)

$$0 \le J_n \le \frac{1}{n+1}$$

But $\lim\limits_{n\to\infty} \frac{1}{n+1} = 0$

So $\lim\limits_{n\to\infty} J_n = 0$

From part (iv)

$$\frac{J_n}{n!} = 1 - \frac{1}{e}\sum_{r=0}^{n} \frac{1}{r!}$$

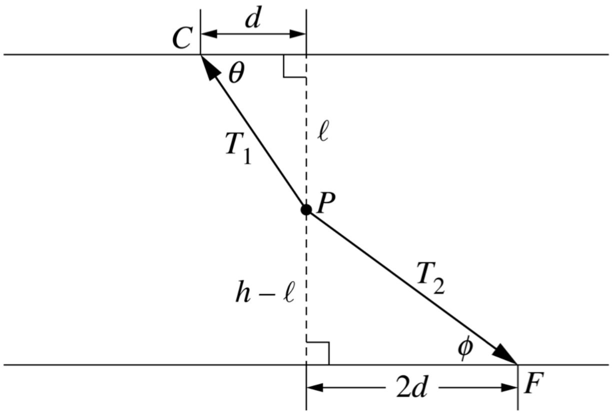
So $\displaystyle\sum_{r=0}^{n} \frac{1}{r!} = e - \frac{e\,J_n}{n!}$

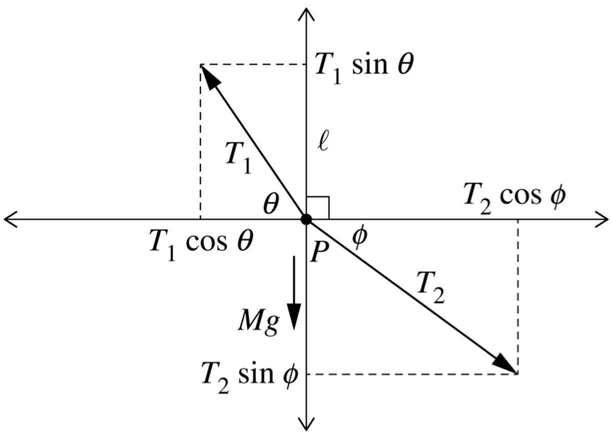
$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}\right) &= \lim_{n\to\infty}\left(e - \frac{e\,J_n}{n!}\right)\\ &= e - 0\\ &= e\end{aligned}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	3
• Obtains equations for the forces in both the horizontal and vertical directions	2
• Obtains an equation for the forces in either the horizontal or vertical direction	1

Sample answer:





Resolving the forces horizontally,

$$T_1 \cos\theta = T_2 \cos\phi \qquad \text{(I)}$$

Resolving the forces vertically,

$$T_1 \sin\theta = Mg + T_2 \sin\phi \qquad \text{(II)}$$

$\frac{\text{(II)}}{\text{(I)}}$ gives, $\tan\theta = \dfrac{Mg + T_2 \sin\phi}{\cos\phi}$

$$\therefore \quad \tan\theta = \frac{Mg}{T_2 \cos\phi} + \tan\phi$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Writes the equation from part (i) in terms of ℓ, d and h OR shows that $\theta > \phi$ OR considers the case where P is $\frac{2h}{3}$ OR equivalent merit	1

Sample answer:

$$\tan\theta = \frac{Mg}{T_2\cos\phi} + \tan\phi$$

From the diagram

$$\tan\theta = \frac{\ell}{d} \quad \text{and} \quad \tan\phi = \frac{h-\ell}{2d}$$

Therefore $$\frac{\ell}{d} = \frac{Mg}{T_2\cos\phi} + \frac{h-\ell}{2d}$$

$$\frac{Mg}{T_2\cos\phi} > 0$$

Thus

$$\frac{\ell}{d} > \frac{h-\ell}{2d}$$

$$\ell > \frac{h-\ell}{2}$$

$$\frac{3\ell}{2} > \frac{h}{2}$$

$$\ell > \frac{h}{3}$$

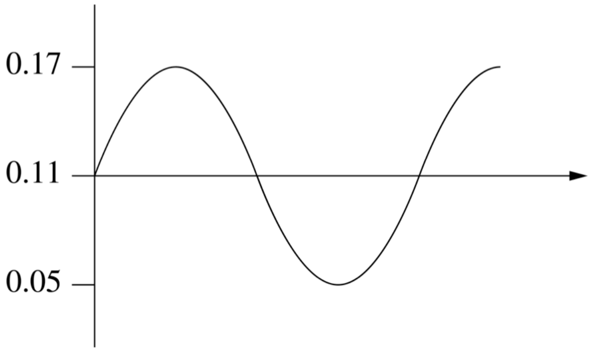
So $h - \ell < \frac{2h}{3}$

$\therefore$ P cannot be lifted to a position $\frac{2h}{3}$ metres above the floor.

Question 15 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains an equation of motion of the piston, or equivalent merit	2
• Finds the period of the motion, or equivalent merit	1

Sample answer:



Amplitude $a = \dfrac{0.17 - 0.05}{2} = 0.06$ m

Period of motion = $\dfrac{1}{40} = 0.025$ s

Equation of motion simple harmonic motion (SHM) is

$\ddot{x} = -n^2(x - b)$

$n = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.025} = 80\pi$ and

Centre of motion $b = 0.11$

$\therefore \quad \ddot{x} = -(80\pi)^2(x - 0.11)$

Acceleration is maximum at an extremity,

$$\begin{aligned}\ddot{x}_{\text{max}} &= -(80\pi)^2(0.05 - 0.11)\\ &= 384\pi^2 \text{ m s}^{-2}\end{aligned}$$

The maximum force on the piston $f = m\ddot{x}$

$= 0.8 \times 384\pi^2$ newtons

≈ 3032 newtons

Question 15 (c)

Criteria	Marks
• Provides correct solution	4
• Attempts to use integration by parts with the correct integral, or equivalent merit	3
• Obtains integral in terms of θ, or equivalent merit	2
• Attempts to use the given substitution, or equivalent merit	1

Sample answer:

Let $x = \tan^2\theta$

at $x = 0 \quad \theta = 0$

at $x = 1 \quad \theta = \dfrac{\pi}{4}$

$$dx = 2\tan\theta\sec^2\theta\, d\theta$$

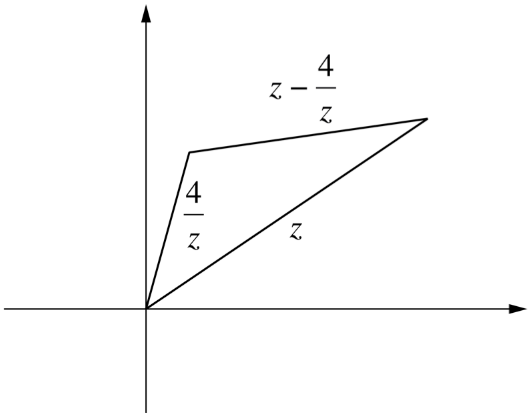
$$\begin{aligned}\sin^{-1}\sqrt{\frac{x}{1+x}} &= \sin^{-1}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}\\ &= \sin^{-1}\sqrt{\sin^2\theta}\\ &= \theta \qquad \text{for } 0 \le \theta \le \frac{\pi}{4}\text{, as } \sin\theta \ge 0\end{aligned}$$

$$\begin{aligned}\int_0^1 \sin^{-1}\sqrt{\frac{x}{1+x}}\,dx &= \int_0^{\frac{\pi}{4}} 2\theta\tan\theta\sec^2\theta\, d\theta \qquad \text{Integration by part}\\ &= \left[\theta\tan^2\theta\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2\theta\, d\theta\\ &= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} -1 + \sec^2\theta\, d\theta\\ &= \frac{\pi}{4} - \left[-\theta + \tan\theta\right]_0^{\frac{\pi}{4}}\\ &= \frac{\pi}{4} + \frac{\pi}{4} - 1\\ &= \frac{\pi}{2} - 1\end{aligned}$$

Question 15 (d)

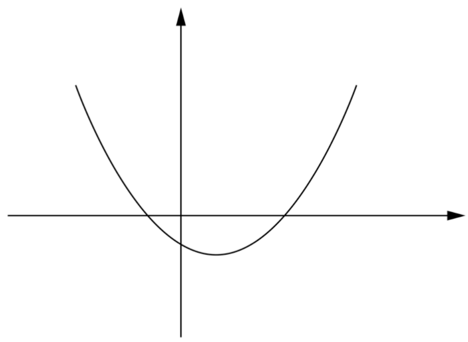
Criteria	Marks
• Provides correct solution	3
• Uses the triangle inequality to obtain a relevant inequality involving only $\lvert z \rvert$, or equivalent merit	2
• Correctly uses the triangle inequality, or equivalent merit	1

Sample answer:



By triangle inequality

$$
\begin{aligned}
|z| &\le \left| z - \frac{4}{z} \right| + \left| \frac{4}{z} \right| \\
|z| &\le 2 + \frac{4}{|z|} \\
|z|^2 &\le 2|z| + 4 \\
|z|^2 - 2|z| - 4 &\le 0 \\
|z| &\le \frac{2 + \sqrt{4 + 4 \times 4}}{2} \\
&= \frac{2 + \sqrt{20}}{2} \\
&= 1 + \sqrt{5}
\end{aligned}
$$



Hence $|z| \le 1 + \sqrt{5}$

Question 16 (a)

Criteria	Marks
• Provides correct solution	4
• Equates real or imaginary parts from $z_C - z_A = e^{i\frac{\pi}{3}}\left(z_B - z_A\right)$ OR Obtains a quadratic in x from $\lvert z_B - z_A \rvert = \lvert z_C - z_A \rvert = \lvert z_B - z_C \rvert$	3
• Attempts to use either $z_C - z_A = e^{i\frac{\pi}{3}}\left(z_B - z_A\right)$, or equivalent merit OR $\lvert z_B - z_A \rvert = \lvert z_C - z_A \rvert = \lvert z_B - z_C \rvert$	2
• Observes that $z_B = x + 5i$, for some real number x, or equivalent merit	1

Sample answer:

Let $z_B = b + 5i$ and $z_C = c - 5i$ where b and c are real numbers.

$$e^{i\frac{\pi}{3}}\left(z_B - z_A\right) = z_C - z_A$$

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left((b-5) + 4i\right) = (c-5) - 6i$$

Equating the imaginary parts, we get

$$2 + \frac{\sqrt{3}}{2}(b-5) = -6$$

$$\frac{\sqrt{3}}{2}(b-5) = -8$$

$$b = 5 - \frac{16}{\sqrt{3}}$$

$$\therefore \quad z_B = \left(5 - \frac{16}{\sqrt{3}}\right) + 5i$$

Question 16 (b)

Criteria	Marks
• Provides correct solution	4
• Obtains the velocity, or equivalent merit	3
• Attempts to solve differential equation, or equivalent merit	2
• Attempts to obtain the force equation	1

Sample answer:

$$M\ddot{y} = -Mg - 0.1Mv_y$$

$$\frac{dv_y}{dt} = -\left(g + 0.1v_y\right)$$

$$\int \frac{dv_y}{g + 0.1v_y} = -\int dt$$

$$10\left[\ln\left|g + 0.1v_y\right|\right] = -t + k$$

When $t = 0$, $v_y = v_0$

$$10\ln\left|g + 0.1v_0\right| = k$$

$$\ln\left|g + 0.1v_y\right| = -\frac{t}{10} + \ln\left|g + 0.1v_0\right|$$

$$\ln\left|\frac{g + 0.1v_y}{g + 0.1v_0}\right| = -\frac{t}{10}$$

$$\ln\frac{g + 0.1v_y}{g + 0.1v_0} = -\frac{t}{10}$$ as $g + 0.1v > 0$ because all speeds are less than 100 m s^{-1}

$$g + 0.1v_y = \left(g + 0.1v_0\right)e^{-\frac{t}{10}}$$

$$0.1v_y = -g + \left(g + 0.1v_0\right)e^{-\frac{t}{10}}$$

$$\therefore\ v_y = 10\left[-g + \left(g + 0.1v_0\right)e^{-\frac{t}{10}}\right]$$

Integrating with respect to t,

$$y = 10\left[-gt + \left(g + 0.1v_0\right)\frac{e^{-\frac{t}{10}}}{-\frac{1}{10}}\right] + c$$

$t = 0, \qquad y = 0$

$$0 = 10\left[-10\left(g + 0.1v_0\right)\right] + c$$

$$\therefore \quad c = 100\left(g + 0.1v_0\right)$$

$$\therefore \quad y = 10\left[-gt - 10\left(g + 0.1v_0\right)e^{-\frac{t}{10}}\right] + 100\left(g + 0.1v_0\right)$$

$t = 7, \qquad y = 0, \qquad g = 10$

$$\begin{aligned} 0 &= 10\left[-70 - 10\left(10 + 0.1v_0\right)e^{-0.7}\right] + 100\left(10 + 0.1v_0\right) \\ &= -700 - 100\left(10 + 0.1v_0\right)e^{-0.7} + 100\left(10 + 0.1v_0\right) \\ 700 &= 100\left[10 + 0.1v_0\right]\left[1 - e^{-0.7}\right] \\ 10 + 0.1v_0 &= \frac{7}{1 - e^{-0.7}} \\ \therefore \quad v_0 &= \left(\frac{7}{1 - e^{-0.7}} - 10\right) \times 10 \\ &= 39.05\ldots \text{ m s}^{-1} \end{aligned}$$

v_0 is 39.1 m s^{-1} correct to 1 decimal place.

Question 16 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to apply given information with three numbers, or equivalent merit	1

Sample answer:

Consider ab, ac, bc

Then $S = 2(ab + ac + bc)$

From the given result

$$ab \times ac \times bc \le \left(\frac{ab + ac + bc}{3}\right)^3$$

$$(abc)^2 \le \left(\frac{S}{6}\right)^3$$

$$abc \le \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

Question 16 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Applies part (i) to ab, ac and bc, or equivalent merit	1

Sample answer:

Given a cube, we have $a = b = c$

So $V = abc = a^3$

And $S = 2(ab + ac + bc) = 6a^2$

$$\left(\frac{S}{6}\right)^{\frac{3}{2}} = \left(\frac{6a^2}{6}\right)^{\frac{3}{2}} = a^3$$

So $V = \left(\frac{S}{6}\right)^{\frac{3}{2}}$ From part (i) we know that $V \le \left(\frac{S}{6}\right)^{\frac{3}{2}}$ for all rectangular prisms

Hence a cube has maximum possible volume for a fixed surface area.

Question 16 (d)

Criteria	Marks
• Provides correct solution	3
• Shows, $z_1z_2 + z_1z_3 + z_2z_3 = 1$	2
• Treats z_1, z_2 and z_3 as zeros of a polynomial OR Shows $\lvert z_1 \rvert = 1$	1

Sample answer:

As $\quad z_1z_2z_3 = 1$

We have $\quad |z_1||z_1||z_3| = 1$

But $\quad |z_1| = |z_2| = |z_3|$

So $\quad |z_1|^3 = 1$

$$|z_1| = 1$$

Hence $\quad |z_1| = |z_2| = |z_3| = 1$

$$z_1 = e^{i\theta}, \quad \text{for some } \theta$$

$$\frac{1}{z_1} = e^{-i\theta} = \overline{e^{i\theta}} = \overline{z_1}$$

Similarly $\quad \dfrac{1}{z_2} = \overline{z_2}$

$$\frac{1}{z_3} = \overline{z_3}$$

$$z_1z_2z_3 = 1$$

So $\quad z_1z_2 = \dfrac{1}{z_3} = \overline{z_3}$

Similarly $\quad z_1z_3 = \overline{z_2}$ and $z_2z_3 = \overline{z_1}$

Hence $\quad z_1z_2 + z_1z_3 + z_2z_3 = \overline{z_3} + \overline{z_2} + \overline{z_1}$

$$= \overline{z_1 + z_2 + z_3}$$

$$= 1$$

Therefore z_1, z_2, z_3 are the zeros of the polynomial

$$z^3 - z^2 + z - 1 = 0$$
$$(z - 1)(z^2 + 1) = 0$$
$$z = 1, \quad i, \quad -i$$

In some order z_1, z_2, z_3 are 1, i, $-i$.

2022 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-N2 Using Complex Numbers	MEX12-4
2	1	MEX-P1 The Nature of Proof	MEX12-2
3	1	MEX-P1 The Nature of Proof	MEX12-8
4	1	MEX-C1 Further Integration	MEX12-5
5	1	MEX-C1 Further Integration	MEX12-5
6	1	MEX-N2 Using Complex Numbers	MEX12-4
7	1	MEX-P1 The Nature of Proof	MEX12-8
8	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
9	1	MEX-V1 Further Work with Vectors	MEX12-3
10	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-6

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	MEX-N1 Introduction to Complex Numbers	MEX12-1
11 (b)	2	MEX-C1 Further Integration	MEX12-5
11 (c) (i)	2	MEX-N1 Introduction to Complex Numbers	MEX12-1
11 (c) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12-1
11 (d)	3	MEX-V1 Further Work with Vectors	MEX12-3
11 (e)	2	MEX-V1 Further Work with Vectors	MEX12-3
11 (f)	3	MEX-C1 Further Integration	MEX12-5
12 (a)	2	MEX-P1 The Nature of Proof	MEX12-2
12 (b)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
12 (c) (i)	1	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
12 (c) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
12 (d)	4	MEX-C1 Further Integration	MEX12-5, MEX12-7
12 (e)	3	MEX-N1 Introduction to Complex Numbers	MEX12-4
13 (a)	3	MEX-P1 The Nature of Proof	MEX12-2, MEX12-8
13 (b)	4	MEX-P2 Further Proof by Mathematical Induction	MEX12-2

Question	Marks	Content	Syllabus outcomes
13 (c) (i)	2	MEX-N2 Using Complex Numbers	MEX12-4
13 (c) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12-4
13 (c) (iii)	3	MEX-N2 Using Complex Numbers	MEX12-4
14 (a) (i)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (a) (ii)	1	MEX-V1 Further Work with Vectors	MEX12-3
14 (a) (iii)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (a) (iv)	2	MEX-V1 Further Work with Vectors	MEX12-3
14 (b) (i)	1	MEX-C1 Further Integration	MEX12-5
14 (b) (ii)	2	MEX-C1 Further Integration	MEX12-5
14 (b) (iii)	2	MEX-C1 Further Integration	MEX12-5
14 (b) (iv)	2	MEX-P2 Further Proof by Mathematical Induction	MEX12-2
14 (b) (v)	1	MEX-P1 The Nature of Proof	MEX12-7
15 (a) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-6
15 (a) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
15 (b)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12-7
15 (c)	4	MEX-C1 Further Integration	MEX12-5
15 (d)	3	MEX-P1 The Nature of Proof MEX-N1 Introduction to Complex Numbers	MEX12-2, MEX12-8
16 (a)	4	MEX-N1 Introduction to Complex Numbers MEX-N2 Using Complex Numbers	MEX12-4, MEX12-7
16 (b)	4	MEX-M1 Applications of Calculus to Mechanics	MEX12-6, MEX12-7
16 (c) (i)	2	MEX-P1 The Nature of Proof	MEX12-7
16 (c) (ii)	2	MEX-P1 The Nature of Proof	MEX12-7
16 (d)	3	MEX-N2 Using Complex Numbers	MEX12-4