2021 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	A
3	B
4	C
5	D
6	A
7	D
8	D
9	B
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Finds the correct argument	1

Sample answer:

$$zw = (2 \times 6)e^{i\left(\frac{\pi}{2}+\frac{\pi}{6}\right)}$$

$$= 12e^{\frac{i2\pi}{3}}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Expands the notation and provides the correct expression with 5 terms OR • Correctly evaluates i^3 or i^4 or i^5	1

Sample answer:

$$\sum_{n=1}^{n}(i)^n = i + (i)^2 + (i)^3 + (i)^4 + (i)^5$$

$$= i - 1 - i + 1 + i$$

$$= i$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Correctly finds the dot product and both magnitudes OR • Uses the correct dot product or a correct magnitude in $a \cdot b = \lvert a \rvert \lvert b \rvert \cos\theta$	2
• Correctly finds the dot product of the two vectors OR • Correctly finds the magnitude of one vector	1

Sample answer:

$$\text{Dot product } \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -6 + 0 + 8$$

$$= 2$$

Magnitudes:

$$\left\lvert \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\rvert = \sqrt{2^2 + 0^2 + 4^2}$$

$$= \sqrt{20}$$

$$\left\lvert \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \right\rvert = \sqrt{(-3)^2 + 1^2 + 2^2}$$

$$= \sqrt{14}$$

$$\therefore \ \sqrt{20}\sqrt{14}\cos\theta = 2$$

$$\cos\theta = 0.1195\ldots$$

$$\theta = 83.135°\ldots$$

$$\approx 83.1° \qquad \text{(1 decimal place)}$$

Question 11 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the square of $(x + iy)$, or equivalent merit OR • Attempts to use the modulus/argument form of $-i$ OR • Plots $-i$ on an Argand diagram and attempts to locate the square roots	1

Sample answer:

Let $(x+iy)^2 = -i$ where $x, y \in \mathbb{R}$

$\therefore x^2 - y^2 = 0$ and $2xy = -1$

$\therefore y = \dfrac{-1}{2x}$

$$\therefore x^2 - \left(\frac{-1}{2x}\right)^2 = 0$$

$$4x^4 - 1 = 0$$

$$x^4 = \frac{1}{4}$$

$$x^2 = \frac{1}{2} \quad (x \in \mathbb{R})$$

$$\therefore \quad x = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ square roots are $\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$ and $\dfrac{-1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

Question 11 (d) (ii)

Criteria	Marks
• Provides correct answers	2
• Uses the quadratic formula to obtain $z = -1 \pm \sqrt{-i}$, or equivalent merit	1

Sample answer:

$$z^2 + 2z + 1 + i = 0$$

$$\begin{aligned} \therefore \quad z &= \frac{-2 \pm \sqrt{(2)^2 - 4(1)(1+i)}}{2} \\ &= \frac{-2 \pm 2\sqrt{1-1-i}}{2} \\ &= -1 \pm \sqrt{-i} \end{aligned}$$

From part (i):

$$z = -1 + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \quad \text{or} \quad z = -1 - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

that is,

$$z = \frac{\left(1-\sqrt{2}\right)}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \quad \text{or} \quad z = \frac{-\left(1+\sqrt{2}\right)}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Identifies $\overline{z}$, or equivalent merit	1

Sample answer:

$z = 5 + i \qquad w = 2 - 4i$

$\therefore \overline{z} = 5 - i$

$$\frac{\overline{z}}{w} = \frac{(5-i)}{(2-4i)} \times \frac{(2+4i)}{(2+4i)}$$

$$= \frac{10 + 20i - 2i + 4}{4 + 16}$$

$$= \frac{14 + 18i}{20}$$

$$= \frac{14}{20} + \frac{18}{20}i$$

$$= \frac{7}{10} + \frac{9}{10}i$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Evaluates one of the three coefficients	2
• Provides a correct expression for the sum of fractions with unknown coefficients, or equivalent merit	1

Sample answer:

Let $\dfrac{3x^2-5}{(x-2)(x^2+x+1)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+x+1}$

$$= \frac{A(x^2+x+1)+(Bx+C)(x-2)}{(x-2)(x^2+x+1)}$$

$\therefore 3x^2-5 = A\left(x^2+x+1\right)+(Bx+C)(x-2)$

when $x=2$

$$3(2)^2-5 = A(2^2+2+1)$$
$$7 = 7A$$
$$A = 1$$

Equating coefficients of x^2

$3 = A+B$

$\therefore B = 2$

Equating constants:

$-5 = A-2C$

$2C = 6$

$C = 3$

$$\therefore \frac{3x^2-5}{(x-2)(x^2+x+1)} = \frac{1}{x-2} + \frac{2x+3}{x^2+x+1}$$

Question 12 (a)

Criteria	Marks
• Provides correct solution	3
• Integrates one of the correct fractions OR • Writes the integrand as the correct sum of fractions and completes the square in a denominator, or equivalent merit	2
• Attempts to write the integrand as a sum of two suitable fractions, or equivalent merit	1

Sample answer:

$$\int \frac{2x+3}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$$= \ln\left(x^2+2x+2\right) + \int \frac{1}{1+(x+1)^2}\,dx + C$$

$$= \ln\left(x^2+2x+2\right) + \tan^{-1}(x+1) + k$$

Question 12 (b) (i)

Criteria	Marks
• States the converse	1

Sample answer:

The converse is 'If n is even, then n^2 is even'.

Question 12 (b) (ii)

Criteria	Marks
• Provides correct proof	1

Sample answer:

If n is even then $n = 2k$ for some integer k, then $n^2 = (2k)^2 = 2(2k^2)$ which is also even.

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Finds the value of p or μ	2
• Evaluates the dot product of the direction vectors OR • Recognises that $\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$ OR • Equates component(s) of $\mathbf{r}_1$ and $\mathbf{r}_2$	1

Sample answer:

$$\mathbf{r}_1 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix} 4 \\ -2 \\ q \end{pmatrix} + \mu \begin{pmatrix} p \\ 3 \\ -1 \end{pmatrix}$$

They are perpendicular $\therefore\ \mathbf{r}_1 \cdot \mathbf{r}_2 = 0$

$$\begin{aligned} 1 \times p + 0 \times 3 + 2 \times -1 &= 0 \\ p - 2 &= 0 \\ p &= 2 \end{aligned}$$

They intersect $\therefore$ components are equal

① $\quad -2 + \lambda = 4 + p\mu$

② $\quad 1 = -2 + 3\mu \qquad \Rightarrow 3 = 3\mu$ so $\mu = 1$

③ $\quad 3 + 2\lambda = q - \mu$

Substitute $\mu = 1$ and $p = 2$ into ①

$$\begin{aligned} -2 + \lambda &= 4 + 2 \times 1 \\ \lambda &= 8 \end{aligned}$$

Substitute $\lambda = 8$ and $\mu = 1$ into ③

$$\begin{aligned} 3 + 2 \times 8 &= q - 1 \\ 19 &= q - 1 \\ q &= 20 \end{aligned}$$

$\therefore\ p = 2 \quad$ and $\quad q = 20.$

Question 12 (d)

Criteria	Marks
• Provides correct solution	3
• Shows that $p(k) \Rightarrow p(k+1)$ is a true statement, or equivalent merit	2
• Establishes initial case, or equivalent merit	1

Sample answer:

$\sqrt{n!} > 2^n, \quad n \geq 9$

Prove it's true for $n = 9$

$$\sqrt{9!} = 602.39\ldots$$
$$2^9 = 512$$
$$\therefore \sqrt{9!} > 2^9$$

$\therefore$ It's true for $n = 9$

Assume it's true for $n = k$: Assume $\sqrt{k!} > 2^k$

Prove it's true for $n = k + 1$: Required to prove $\sqrt{(k+1)!} > 2^{k+1}$

$$\begin{aligned} \text{LHS} &= \sqrt{(k+1)!} \\ &= \sqrt{k!} \times \sqrt{(k+1)} \\ &> 2^k \times \sqrt{(k+1)} && \text{using the assumption} \\ &> 2^k \times 2 && \text{because } \sqrt{k+1} \text{ is greater than 2 for } k \geq 9 \\ &= 2^{k+1} \\ &= \text{RHS} \end{aligned}$$

$\therefore \sqrt{(k+1)!} > 2^{k+1}$ as required

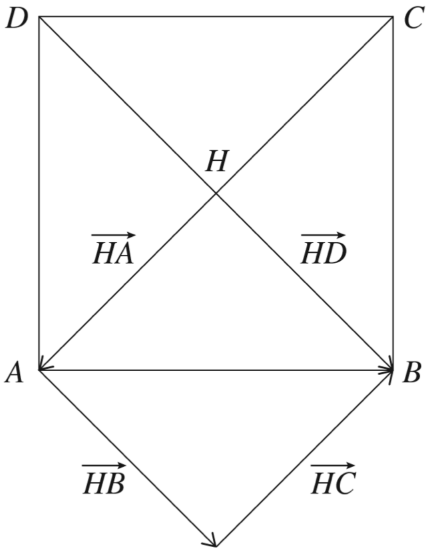
$\therefore$ By principle of mathematical induction, the inequality is true for $n \geq 9$.

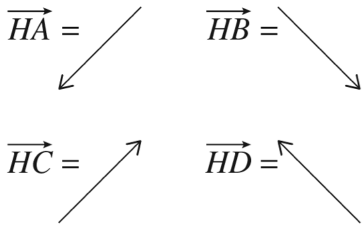
Question 12 (e) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Consider the square base





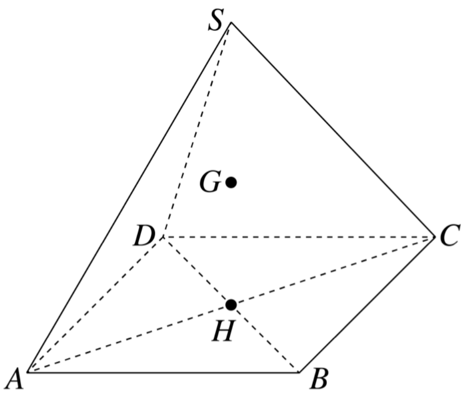
As H is the midpoint of AC and BD, when you add them head to tail, you end up back where you started.

$$\therefore\ \overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \underset{\sim}{0}$$

Question 12 (e) (ii)

Criteria	Marks
• Provides correct solution	2
• Writes $\overrightarrow{GA} = \overrightarrow{GH} + \overrightarrow{HA}$, or equivalent merit	1

Sample answer:



$$\overrightarrow{GA} = \overrightarrow{GH} + \overrightarrow{HA}$$
$$\overrightarrow{GB} = \overrightarrow{GH} + \overrightarrow{HB}$$
$$\overrightarrow{GC} = \overrightarrow{GH} + \overrightarrow{HC}$$
$$\overrightarrow{GD} = \overrightarrow{GH} + \overrightarrow{HD}$$

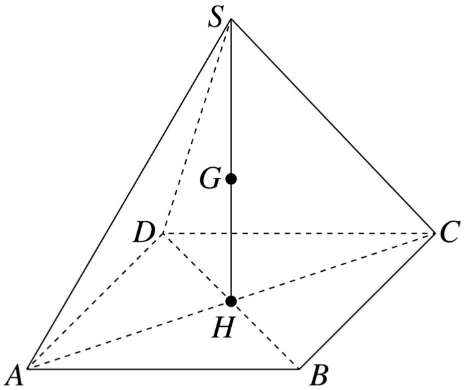
Adding these

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = 4\overrightarrow{GH} + \overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD}$$
$$-\overrightarrow{GS} = 4\overrightarrow{GH} + \underset{\sim}{0}$$
$$\therefore\ 4\overrightarrow{GH} + \overrightarrow{GS} = \underset{\sim}{0}$$

Question 12 (e) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:



From part (ii)

$$4\overrightarrow{GH} + \overrightarrow{GS} = \underset{\sim}{0}$$

$$4\overrightarrow{GH} + \overrightarrow{GH} + \overrightarrow{HS} = \underset{\sim}{0}$$

$$5\overrightarrow{GH} + \overrightarrow{HS} = \underset{\sim}{0}$$

$$\overrightarrow{HS} = -5\overrightarrow{GH}$$

$$= 5\overrightarrow{HG}$$

$$\overrightarrow{HG} = \frac{1}{5}\overrightarrow{HS}$$

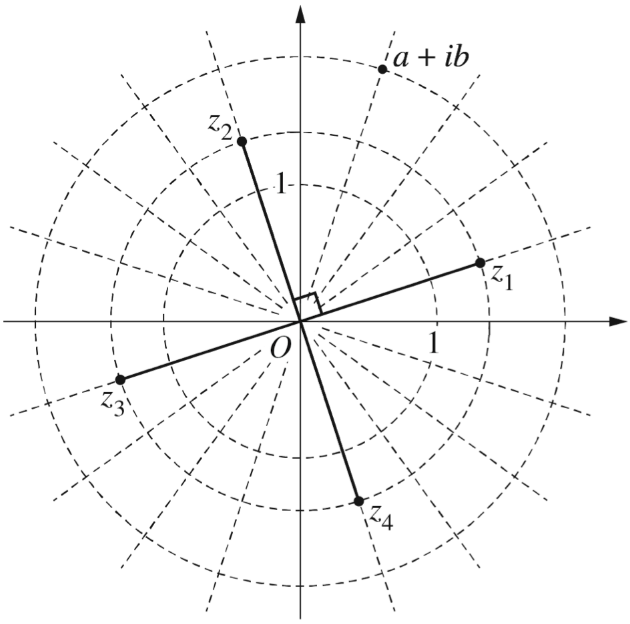
$$\therefore \lambda = \frac{1}{5}$$

Question 13 (a)

Criteria	Marks
• Provides appropriate sketch	2
• Indicates one correct argument OR • Indicates correct modulus	1

Sample answer:

z_1, z_2, z_3 and z_4 represent the fourth roots.



Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains correct primitive OR correct definite integral in terms of u, or equivalent merit	2
• Attempts to use a suitable substitution	1

Sample answer:

$$I = \int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx$$

$$= \int_{\sqrt{10}}^{\sqrt{13}} x^2\left(x^2-9\right)\frac{x}{\sqrt{x^2-9}}\,dx$$

$$= \int_1^2 \left(u^2+9\right)u^2\,du$$

$$= \int_1^2 u^4+9u^2\,du$$

$$= \left[\frac{u^5}{5}+3u^3\right]_1^2$$

$$= \frac{2^5}{5}+3\times 2^3-\left(\frac{1}{5}+3\right)$$

$$= \frac{32}{5}-\frac{1}{5}+24-3$$

$$= \frac{31}{5}+21$$

$$= \frac{136}{5}$$

Let $u = \sqrt{x^2-9}$

$$du = \frac{x}{\sqrt{x^2-9}}\,dx$$

and $u^2 = x^2-9$

When $x=\sqrt{10}$, $u=\sqrt{10-9}=1$

When $x=\sqrt{13}$, $u=\sqrt{13-9}=2$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Uses integration by parts, or equivalent	1

Sample answer:

$$I_n = \int_1^e (\ln x)^n \, dx, \qquad n \geq 0$$

Let $\quad u = (\ln x)^n \qquad$ and $\quad v' = 1$

$$\therefore \quad u' = \frac{n(\ln x)^{n-1}}{x} \qquad\qquad v = x$$

$$\therefore I_n = \Big[x(\ln x)^n \Big]_1^e - \int_1^e n(\ln x)^{n-1} \, dx$$

$$= e(\ln e)^n - (\ln 1)^n - nI_{n-1}$$

$$= e - 0 - nI_{n-1}$$

$$= e - nI_{n-1}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	4
• Shows $V_B = \pi$, or equivalent	3
• Obtains volume V_A • Writes volume V_A or V_B as a difference of volumes involving an integral expression	2
• States volume of a relevant sphere OR hemisphere OR cylinder OR • Provides correct integral expression for a relevant volume	1

Sample answer:

$$V_A = \pi \int_0^1 (1)^2 - \left(1 - x^2\right) dx$$

$$= \pi \int_0^1 x^2\, dx$$

$$= \pi \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{\pi}{3}$$

$$V_B = \pi \int_1^e (1)^2 - (\ln x)^2\, dx$$

$$= \pi \Big[x \Big]_1^e - \pi \int_1^e (\ln x)^2\, dx$$

$$= \pi(e-1) - \pi I_2 \qquad \text{from part (i)}$$

Now $I_2 = e - 2I_1$

$$I_1 = e - I_0$$

$$I_0 = \int_1^e 1\, dx$$

$$= \Big[x \Big]_1^e$$

$$= e - 1$$

$$\therefore I_2 = e - 2\big[e - (e-1)\big] = e - 2$$

$$\therefore V_B = \pi(e-1) - \pi(e-2)$$

$$= -\pi + 2\pi$$

$$= \pi$$

$$\therefore V_A : V_B = \frac{\pi}{3} : \pi$$

$$= 1 : 3$$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	2
• States the amplitude of the motion, or equivalent merit	1

Sample answer:

$\ddot{x} = -4(x-3)$

$\therefore n^2 = 4, \quad \text{centre is } x = 3$

$n = 2$

$\text{period} = \frac{2\pi}{2} = \pi$

$v^2 = n^2\left(a^2 - (x-c)^2\right)$

when $\quad v = 8, x = 0$

$64 = 4\left(a^2 - (0-3)^2\right)$

$16 = a^2 - 9$

$a^2 = 25$

$a = 5$

$\therefore$ The particle oscillates between

$x = 3+5 \quad \text{and} \quad x = 3-5$

$= 8 \qquad\qquad = -2$

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds the displacement function, or equivalent merit	1

Sample answer:

Consider $x = a\cos(nt+\alpha)+c$, where

$a = 5, \quad n = 2, \quad c = 3$

$\therefore\ x = 5\cos(2t+\alpha)+3$

When $\quad t = 0, \quad x = 5.5$

$$5.5 = 5\cos\alpha + 3$$

$$\frac{2.5}{5} = \cos\alpha$$

$$\therefore\quad \alpha = \cos^{-1}\left(\frac{2.5}{5}\right)$$

$$= 1.04719\ldots\text{radians}$$

$$0 = 5\cos(2t+1.04719\ldots)+3$$

$$\cos(2t+1.04719\ldots) = \frac{-3}{5}$$

$$2t+1.04719\ldots = 2.214\ldots\text{radians}$$

$$2t = 1.167\ldots$$

$$t = 0.583\ldots$$

$\therefore$ First value of t when $x = 0$ is $t = 0.58$ seconds (2 decimal places).

Question 14 (a)

Criteria	Marks
• Provides correct solution	4
• Applies partial fraction to the correct integral, or equivalent merit	3
• Completes t-substitution, including the limits of integration, or equivalent merit	2
• Attempts to use a t-substitution, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx \qquad \text{let } t = \tan\frac{x}{2} \quad \therefore \cos x = \frac{1-t^2}{1+t^2}$$

when $x = 0, \quad t = 0$

$x = \dfrac{\pi}{2}, \quad t = 1$

$$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(1+t^2\right)dx$$

$$\therefore dx = \frac{2}{1+t^2}\,dt$$

$$\therefore \text{Integral} = \int_0^1 \frac{1}{3+\frac{5\left(1-t^2\right)}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{3+3t^2+5-5t^2}\,dt$$

$$= \int_0^1 \frac{2}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{1}{4-t^2}\,dt$$

$$= \int_0^1 \frac{1}{(2-t)(2+t)}\,dt$$

$$= \frac{1}{4}\int_0^1 \frac{1}{2-t} + \frac{1}{2+t}\,dt$$

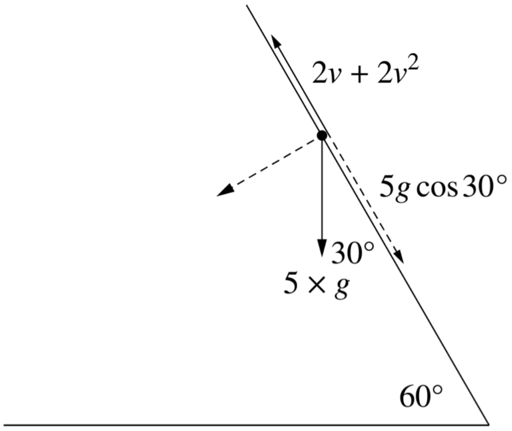
$$= \frac{1}{4}\left[\ln\left|\frac{2+t}{2-t}\right|\right]_0^1$$

$$= \frac{1}{4}(\ln 3 - \ln 1) = \frac{1}{4}\ln 3$$

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Finds appropriate component of force due to gravity OR • Provides appropriate diagram, or equivalent merit	1

Sample answer:



Resultant force down slope $= 5g\cos 30° - 2v - 2v^2$

$$= \frac{5\sqrt{3}}{2}g - 2v - 2v^2$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• States that the resultant force is $\underset{\sim}{0}$, or equivalent merit	1

Sample answer:

Constant speed $\Rightarrow$ Resultant force = 0

$$\therefore \frac{5\sqrt{3}}{2} g = 2v + 2v^2$$

$$g = 10 \quad \therefore 25\sqrt{3} = 2v + 2v^2$$

$$2v^2 + 2v - 25\sqrt{3} = 0$$

$$\therefore v = \frac{-2 \pm \sqrt{4 + 4(2)\left(25\sqrt{3}\right)}}{4}$$

$$= 4.1798\ldots \text{ or } -5.1798\ldots$$

$$\therefore \text{speed} = 4.1798\ldots \text{ as moving down the slope}$$

$$\simeq 4.2 \text{ m s}^{-1} \quad \text{(1 decimal place)}$$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Expands using binomial theorem AND • Expands using De Moivre's theorem	2
• Expands using binomial theorem OR • Expands using De Moivre's theorem	1

Sample answer:

Using De Moivre:

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

Using binomial expansion

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 = {} & (\cos\theta)^5 + 5(\cos\theta)^4(i\sin\theta) \\ & +10(\cos\theta)^3(i\sin\theta)^2 \\ & +10(\cos\theta)^2(i\sin\theta)^3 \\ & +5(\cos\theta)(i\sin\theta)^4 \\ & +(i\sin\theta)^5 \\ = {} & \cos^5\theta + 5\cos^4\theta\sin\theta i \\ & -10\cos^3\theta\sin^2\theta \\ & -10\cos^2\theta\sin^3\theta i \\ & +5\cos\theta\sin^4\theta \\ & +\sin^5\theta i\end{aligned}$$

Equating real parts:

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2 \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\left(1-2\cos^2\theta+\cos^4\theta\right) \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds the solutions to the degree 5 polynomial, or equivalent merit	2
• Uses $\theta = \frac{\pi}{10}$ in the equation from part (i), or equivalent merit	1

Sample answer:

$$\operatorname{Re}\left(e^{i\frac{\pi}{10}}\right) = \operatorname{Re}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$
$$= \cos\frac{\pi}{10}$$

When $\theta = \frac{\pi}{10}$, $\cos 5\theta = \cos\frac{\pi}{2} = 0$

$\therefore \cos\frac{\pi}{10}$ is a root of $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$$
$$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) = 0$$
$$\cos\frac{\pi}{10} \neq 0$$

$\therefore \cos\frac{\pi}{10}$ is a root of $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$$\therefore \cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$$
$$= \frac{20 \pm 4\sqrt{5}}{32}$$
$$= \frac{5 \pm \sqrt{5}}{8}$$
$$\therefore \cos\frac{\pi}{10} = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}$$

But roots of $\cos 5\theta = 0$ are:

$$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$$
$$\therefore \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$$

so roots of $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ are $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{\pi}{2} = 0, \cos\frac{7\pi}{10}$ and $\cos\frac{9\pi}{10}$

From the graph of $y = \cos x$, we see that $\cos\frac{\pi}{10}$ is the largest of these.

$$\therefore \cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$$

$$\therefore \mathrm{Re}\left(e^{i\frac{\pi}{10}}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use appropriate expressions for x and y	1

Sample answer:

$$\sqrt{abc} = \sqrt{(ab)c} \le \frac{ab+c}{2} \qquad \text{using the result provided}$$

$$= \frac{\sqrt{a^2b^2}+c}{2}$$

$$\le \frac{\frac{a^2+b^2}{2}+c}{2} \qquad \text{using the same result a second time}$$

$$= \frac{a^2+b^2+2c}{4}$$

Answers could include:

Let $x = \dfrac{a^2+b^2}{2}$ and $y = c$

$$\therefore \sqrt{\frac{a^2+b^2}{2}\cdot c} \le \frac{\frac{a^2+b^2}{2}+c}{2}$$

$$\sqrt{\frac{a^2c+b^2c}{2}} \le \frac{a^2+b^2+2c}{4}$$

Let $m = a^2c$ and $n = b^2c$

$$\therefore \sqrt{mn} = \sqrt{a^2c\cdot b^2c} = \sqrt{a^2b^2c^2} = abc$$

$$\sqrt{\frac{m+n}{2}} = \frac{a^2c+b^2c}{2}$$

$$\therefore abc \le \frac{a^2c+b^2c}{2}$$

$$\therefore \sqrt{abc} \le \sqrt{\frac{a^2c+b^2c}{2}} \le \frac{a^2+b^2+2c}{4}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Permutes the result in (i), or equivalent merit	1

Sample answer:

$$\sqrt{abc} \le \frac{a^2 + b^2 + 2c}{4}$$

$$\sqrt{acb} \le \frac{a^2 + c^2 + 2b}{4}$$

$$\sqrt{bca} \le \frac{b^2 + c^2 + 2a}{4}$$

Adding

$$3\sqrt{abc} \le \frac{2a^2 + 2b^2 + 2c^2 + 2a + 2b + 2c}{4}$$

$$3\sqrt{abc} \le \frac{a^2 + b^2 + c^2 + a + b + c}{2}$$

$$\therefore \sqrt{abc} \le \frac{a^2 + b^2 + c^2 + a + b + c}{6} \quad \text{as required}$$

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to use an expression for an odd number in the formula for t_n, or equivalent merit	1

Sample answer:

$$t_n = \frac{n(n+1)}{2} \qquad h_n = 2n^2 - n$$

The odd numbers 1, 3, 5, … can be expressed as $2m - 1$ where m is an integer.
$\therefore$ the odd triangular numbers are t_{2m-1}

$$\begin{aligned} t_{2m-1} &= \frac{(2m-1)(2m-1+1)}{2} \\ &= \frac{(2m-1)2m}{2} \\ &= m(2m-1) \\ &= 2m^2 - m \\ &= h_m \end{aligned}$$

$\therefore$ The odd triangular numbers are hexagonal.

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The even numbers 2, 4, 6 … can be expressed as $2m$ where m is an integer.

$$\begin{aligned} t_{2m} &= \frac{2m(2m+1)}{2} \\ &= m(2m+1) \\ &= 2m^2 + m \end{aligned}$$

Use proof by contradiction to show it is not hexagonal. Assume it is hexagonal.

$$\begin{aligned} 2m^2 + m &= 2k^2 - k && \text{where } k \text{ is an integer} \\ m + k &= 2k^2 - 2m^2 \\ m + k &= 2\left(k^2 - m^2\right) \\ m + k &= 2(k-m)(k+m) \\ \therefore\ 1 &= 2(k-m) && \text{since } k + m \neq 0 \end{aligned}$$

So, 1 is even, which is not true $\therefore$ the original statement is false.

Question 15 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain an expression for t in terms of v, or equivalent merit	2
• Provides an expression for the resultant force, or equivalent merit	1

Sample answer:

$$\text{Force} = -mg - kv^2$$

$$ma = -mg - kv^2$$

$$a = -g - \frac{kv^2}{m}$$

$$a = -g - kv^2 \qquad \text{given } m = 1$$

$$\frac{dv}{dt} = -g - kv^2 \qquad \text{need } t \text{ in terms of } v\text{, so use } a = \frac{dv}{dt}$$

$$\frac{dt}{dv} = -\frac{1}{g + kv^2}$$

$$t = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) + c$$

When $t = 0$, $v = u$

$$0 = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) + c$$

$$c = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

$$\therefore \quad t = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \frac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right)$$

Max height, $v = 0 \therefore t = \dfrac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\dfrac{k}{g}}\right)$ as required.

(Either notation, arctan or $\tan^{-1}$, is acceptable.)

Question 15 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain an expression for x in terms of v, or equivalent merit	2
• Provides an integral expression for x in terms of v, or equivalent merit	1

Sample answer:

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$x = -\frac{1}{2k}\ln\left(g + kv^2\right) + c$$

When $x = 0,\ v = u$

$$0 = -\frac{1}{2k}\ln\left(g + ku^2\right) + c$$

$$c = \frac{1}{2k}\ln\left(g + ku^2\right)$$

$$\therefore\ x = \frac{1}{2k}\left(\ln\left(g + ku^2\right) - \ln\left(g + kv^2\right)\right) = \frac{1}{2k}\ln\frac{g + ku^2}{g + kv^2}$$

Maximum height, sub in $v = 0$

$\therefore$ maximum height $x = \dfrac{1}{2k}\ln\left(\dfrac{g + ku^2}{g}\right)$

Question 15 (d)

Criteria	Marks
• Provides correct solution	2
• Writes $5^n = (2+3)^n$ or equivalent merit	1

Sample answer:

$$5 = 2 + 3$$

so $\quad 5^n = (2+3)^n$

$$= 2^n + 3^n + \binom{n}{1}2 \times 3^{n-1} + \text{other terms}$$

$$> 2^n + 3^n$$

so $\quad 2^n + 3^n \neq 5^n \qquad$ if $n \geq 2$

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	2
• States that $\overrightarrow{OP}$ is a unit vector, or equivalent merit.	1

Sample answer:

Point P is on the unit sphere so

$$
\begin{aligned}
1 &= \left|\overrightarrow{OP}\right| \\
&= \left| x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \right| \\
&\le \left| x\underset{\sim}{i} \right| + \left| y\underset{\sim}{j} + z\underset{\sim}{k} \right| && \text{(triangular inequality)} \\
&\le \left| x\underset{\sim}{i} \right| + \left| y\underset{\sim}{j} \right| + \left| z\underset{\sim}{k} \right| && \text{(triangular inequality)} \\
&= |x| + |y| + |z| && \underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k}\ \text{unit vectors}
\end{aligned}
$$

so $|x| + |y| + |z| \ge 1$

Answers could include:

P is on the unit sphere $\therefore \sqrt{x^2 + y^2 + z^2} = 1$

$\therefore |x|^2 + |y|^2 + |z|^2 = 1$

But $|x| \le 1$, $|y| \le 1$ and $|z| \le 1$

$\therefore |x|^2 \le |x|$, $|y|^2 \le |y|$ and $|z|^2 \le |z|$

$$
\begin{aligned}
\therefore 1 &= |x|^2 + |y|^2 + |z|^2 \\
&\le |x| + |y| + |z|
\end{aligned}
$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains $\left\|a_1b_1 + a_2b_2 + a_3b_3\right\| = \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\cos\theta \ldots$	2
• Attempts to apply the dot product, or equivalent merit	1

Sample answer:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$a_1b_1 + a_2b_2 + a_3b_3 = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$-1 \le \cos\theta \le 1$$

$$-|\underset{\sim}{a}||\underset{\sim}{b}| \le a_1b_1 + a_2b_2 + a_3b_3 \le |\underset{\sim}{a}||\underset{\sim}{b}|$$

so $$\left|a_1b_1 + a_2b_2 + a_3b_3\right| \le |\underset{\sim}{a}||\underset{\sim}{b}|$$

$$= \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$$

Question 16 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Chooses one suitable vector to use with the result from part (ii), or equivalent merit	1

Sample answer:

If $P(x,y,z)$ is on the unit sphere $x^2+y^2+z^2=1$, $|x|^2+|y|^2+|z|^2=1$.
Hence $Q(|x|,|y|,|z|)$ is on the unit sphere.

In part (ii) let $\underset{\sim}{a}=\begin{pmatrix}|x|\\|y|\\|z|\end{pmatrix}$

And $\underset{\sim}{b}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

Then

$$\begin{aligned}\big||x|+|y|+|z|\big| &\le \sqrt{|x|^2+|y|^2+|z|^2}\sqrt{1^2+1^2+1^2}\\ &=\sqrt{3}\end{aligned}$$

$|x|,|y|$ and $|z|$ are not negative, so
$|x|+|y|+|z|=\big||x|+|y|+|z|\big|\le\sqrt{3}$

Answers could include:

$P(x,y,z)$ on unit sphere so $x^2+y^2+z^2=1$.
Choose b_1 to be 1 or -1 so that $xb_1\ge 0$ so $xb_1=|x|$, and similarly choose b_2 and b_3

so $yb_2\ge 0$ and $zb_3\ge 0$.

Let $\underset{\sim}{a}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

By part (ii)

$$\begin{aligned}\left|xb_1+yb_2+zb_3\right| &\le \sqrt{x^2+y^2+z^2}\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}\\ &=1\times\sqrt{1+1+1}\\ &=\sqrt{3}\end{aligned}$$

$\big||x|+|y|+|z|\big|\le\sqrt{3}$

$|x|,|y|,|z|$ are not negative so

$|x|+|y|+|z|\le\sqrt{3}$

Question 16 (b)

Criteria	Marks
• Provides correct solution, with justification	5
• Shows that the relevant times are positive OR • Shows that the relevant times occur before the particle lands, or equivalent merit	4
• Obtains a quadratic that will identify all possible angles, or equivalent merit	3
• Uses the fact that the dot product of the two vectors is 0, or equivalent merit	2
• Obtains velocity vector, or equivalent merit	1

Sample answer:

Position $\quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta \end{pmatrix}$

Velocity $\quad \dot{\underset{\sim}{r}}(t) = \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix}$

If position vector is perpendicular to velocity vector then

$$
\begin{aligned}
0 &= \underset{\sim}{r}(t) \cdot \dot{\underset{\sim}{r}}(t) \\
&= \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta \end{pmatrix} \cdot \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix} \\
&= u^2 t\cos^2\theta + \left(-\frac{gt^2}{2} + ut\sin\theta\right)(-gt + u\sin\theta) \\
&= u^2 t\cos^2\theta + \frac{g^2t^3}{2} - \frac{gt^2u}{2}\sin\theta - gt^2u\sin\theta + u^2t\sin^2\theta \\
&= u^2 t - \frac{3ugt^2\sin\theta}{2} + \frac{g^2t^3}{2} \\
&= \frac{t}{2}\left(g^2t^2 - 3ugt\sin\theta + 2u^2\right)
\end{aligned}
$$

During time of flight $t > 0$ so above can only be zero when $g^2t^2 - 3ugt\sin\theta + 2u^2 = 0$.

$y = g^2t^2 - 3ugt\sin\theta + 2u^2$ is the graph of a concave up parabola where y is a function of t.

Want two zeros so $\triangle = b^2 - 4ac > 0$ and we also know $0 < \theta < \dfrac{\pi}{2}$

$$9u^2g^2\sin^2\theta - 4 \times g^2 \times 2u^2 > 0$$

$$u^2g^2\left(9\sin^2\theta - 8\right) > 0$$

$$9\sin^2\theta - 8 > 0 \qquad \text{as } u^2g^2 > 0$$

$$\sin^2\theta > \frac{8}{9}$$

$$\sin\theta > \frac{\sqrt{8}}{3} \qquad \left(0 < \theta < \frac{\pi}{2} \text{ so } \sin\theta > 0\right)$$

$$\theta > 1.23 \qquad \text{(2 decimal places)}$$

(70.52°)

This shows that we may have two points during the time of flight if $\dfrac{\pi}{2} > \theta > 1.23$

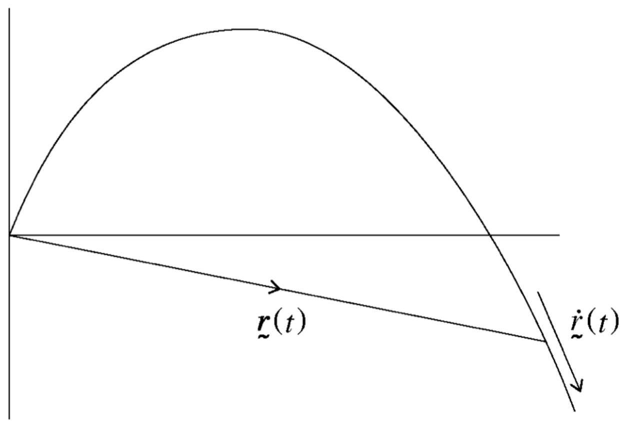
$$t = \frac{3ug\sin\theta \pm ug\sqrt{9\sin^2\theta - 8}}{2g^2} = \frac{u}{2g}\left(3\sin\theta \pm \sqrt{9\sin^2\theta - 8}\right)$$

$$9\sin^2\theta - 8 < 9\sin^2\theta = (3\sin\theta)^2$$

so $t > 0$

so both points occur after projection.

If we ignore the ground, and consider points on the trajectory below the point of projection.



Both $\underset{\sim}{r}(t)$ and $\dot{\underset{\sim}{r}}(t)$ point into the 4th quadrant so the angle between them is less than $\dfrac{\pi}{2}$.

Thus the two points must occur after projection but before the projectile lands.

Question 16 (c)

Criteria	Marks
• Provides correct sketch	3
• Considers inequalities involving $x\tan(x)$ AND has included or excluded at least one section of the Argand plane • Obtains the region below <diag>	2
• States that a nominated section of the Argand plane is included in the required region OR • States that a nominated section of the Argand plane is excluded from the required region	1

Sample answer:

If $z = x + iy$ and $Arg(z) = \theta$

then $-\pi < \theta \le \pi$

$\tan\theta = \dfrac{y}{x}$ when $x \ne 0$

$\text{Re}(z) = x$

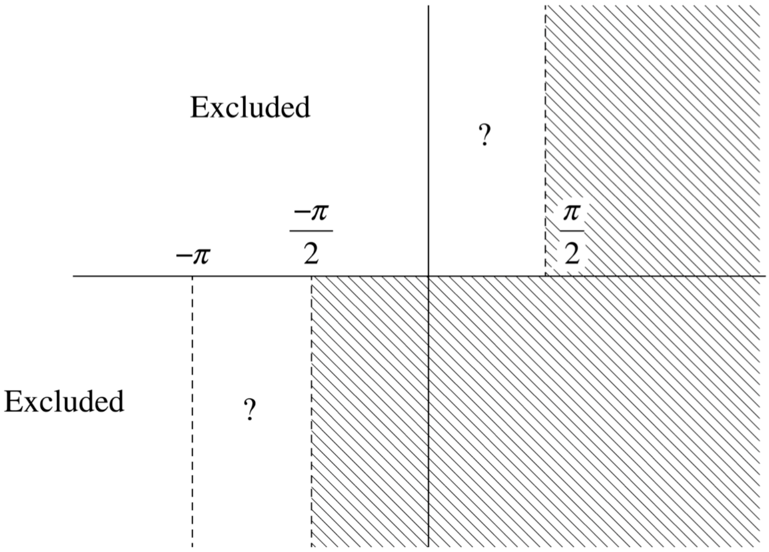
In 2nd quadrant $\text{Re}(z) = x < 0 < \dfrac{\pi}{2} < Arg(z)$ so 2nd quadrant not in region

In 4th quadrant $\text{Re}(z) > 0 > Arg(z)$ so 4th quadrant included in region

In 1st quadrant $Arg(z) < \dfrac{\pi}{2}$ so if $x \ge \dfrac{\pi}{2}$ then z is included in region

In 3rd quadrant $-\pi < Arg(z) < -\dfrac{\pi}{2}$ so $x < -\pi$ excluded and $x > -\dfrac{\pi}{2}$ included.

So far



If $0 \le x < \dfrac{\pi}{2}$ then $\tan x$ is an increasing function

$\text{Re}(z) \ge Arg(z)$

$x \ge \theta$

$\tan x \ge \tan\theta = \dfrac{y}{x} \quad \left(\text{tan increases on} \left[0, \dfrac{\pi}{2}\right]\right)$

$x \tan x \ge y \qquad (x > 0)$

If $-\pi < x < \dfrac{-\pi}{2}$ tan also increasing

$Arg(z) = \arctan\left(\dfrac{y}{x}\right) - \pi$

$x \ge \arctan\left(\dfrac{y}{x}\right) - \pi$

$x + \pi \ge \arctan\left(\dfrac{y}{x}\right)$

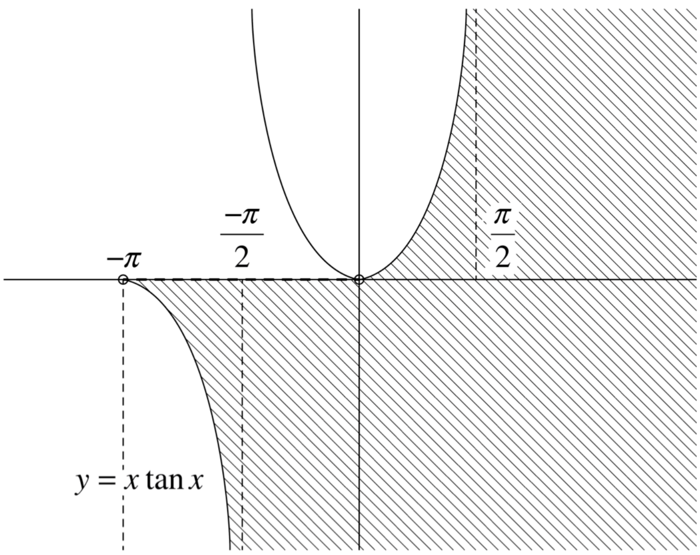
$\tan(x + \pi) \ge \dfrac{y}{x}$

$x \tan(x) \le y$

as $\tan(x + \pi) = \tan x$ and $x < 0$

Also, if $z = 0$ and if $y = 0, x < 0, Arg(z) = \pi > x$

$\therefore$ not included.



2021 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX V1 Further work with vectors	MEX 12 3
2	1	MEX C1 Further integration	MEX 12–5
3	1	MEX V1 Further work with vectors	MEX 12–3
4	1	MEX P1 The nature of proof	MEX 12–2
5	1	MEX P1 The nature of proof	MEX 12–1
6	1	MEX N2 Using complex numbers	MEX 12–4
7	1	MEX V1 Further work with vectors	MEX 12–3
8	1	MEX M1 Applications of calculus to mechanics	MEX 12–6
9	1	MEX P1 The nature of proof	MEX 12–2
10	1	MEX N2 Using complex numbers	MEX 12–4

Section II

Question	Marks	Content	Syllabus outcomes
11(a)	2	MEX N1 Introduction to complex numbers	MEX 12–4
11 (b)	2	MEX P2 Further proof by mathematical induction MEX N1 Introduction to complex numbers	MEX 12–1
11 (c)	3	MEX V1 Further work with vectors	MEX 12–3
11 (d) (i)	2	MEX N2 Using complex numbers	MEX 12–4
11 (d) (ii)	2	MEX N2 Using complex numbers	MEX 12–4
11 (e)	2	MEX N1 Introduction to complex numbers	MEX 12–4
11 (f)	3	MEX C1 Further integration	MEX 12–1
12 (a)	3	MEX C1 Further integration	MEX 12–5
12 (b) (i)	1	MEX P1 The nature of proof	MEX 12–2
12 (b) (ii)	1	MEX P1 The nature of proof	MEX 12–2
12 (c)	3	MEX V1 Further work with vectors	MEX 12–3
12 (d)	3	MEX P2 Further proof by mathematical induction	MEX 12–2
12 (e) (i)	1	MEX V1 Further work with vectors	MEX 12–3
12 (e) (ii)	2	MEX V1 Further work with vectors	MEX 12–3

Question	Marks	Content	Syllabus outcomes
12 (e) (iii)	1	MEX V1 Further work with vectors	MEX 12–3
13 (a)	2	MEX N2 Using complex numbers	MEX 12–4
13 (b)	3	MEX C1 Further integration	MEX 12–5
13 (c) (i)	2	MEX C1 Further integration	MEX 12–5
13 (c) (ii)	4	MEX C1 Further integration	MEX 12–5
13 (d) (i)	2	MEX M1 Application of calculus to mechanics	MEX 12–6
13 (d) (ii)	2	MEX M1 Application of calculus to mechanics	MEX 12–6, MEX 12–7
14 (a)	4	MEX C1 Further integration	MEX 12–5
14 (b) (i)	2	MEX M1 Application of calculus to mechanics	MEX 12–6
14 (b) (ii)	2	MEX M1 Application of calculus to mechanics	MEX 12–6, MEX 12–7
14 (c) (i)	3	MEX N2 Using complex numbers	MEX 12–4
14 (c) (ii)	3	MEX N2 Using complex numbers	MEX 12–4
15 (a) (i)	2	MEX P1 The nature of proof	MEX 12–2
15 (a) (ii)	2	MEX P1 The nature of proof	MEX 12–2
15 (b) (i)	2	MEX P1 The nature of proof	MEX 12–1, MEX 12–2
15 (b) (ii)	1	MEX P1 The nature of proof	MEX 12–1, MEX 12–2
15 (c) (i)	3	MEX M1 Application of calculus to mechanics	MEX 12–6
15 (c) (ii)	3	MEX M1 Application of calculus to mechanics	MEX 12–6
15 (d)	2	MEX P1 The nature of proof	MEX 12–2
16 (a) (i)	2	MEX P1 The nature of proof	MEX 12–2
16 (a) (ii)	3	MEX V1 Further work with vectors	MEX 12–3
16 (a) (iii)	2	MEX V1 Further work with vectors	MEX 12–3
16 (b)	5	MEX M1 Application of calculus to mechanics	MEX 12–6, MEX 12–7
16 (c)	3	MEX N1 Introduction to complex numbers	MEX 12–4