2020 HSC Mathematics Extension 2
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	B
3	C
4	A
5	D
6	A
7	D
8	B
9	C
10	B

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$w = -1 + 4i$$
$$|w| = \sqrt{17}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains $\overline{z}$ OR • Correctly evaluates wz	1

Sample answer:

$$\begin{aligned} w\overline{z} &= (-1+4i)(2+i) \\ &= -2-4+8i-i \\ &= -6+7i \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Correctly identifies the two functions to be used in integration by parts, or equivalent merit	1

Sample answer:

$$\int_1^e x \ln x \, dx$$

$$u = \ln x, \qquad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x}, \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x}{2} \, dx$$

$$= \frac{e^2}{2} - \frac{x^2}{4} \Big|_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	3
• Correctly separates variables, or equivalent merit	2
• Uses $a = v\dfrac{dv}{dx}$, or equivalent merit	1

Sample answer:

$$a = v^2 + v$$

$$\therefore \quad v \times \frac{dv}{dx} = v^2 + v$$

$$\therefore \quad \frac{dx}{dv} = \frac{1}{v+1}$$

$$x = \ln(v+1) + C$$

When $x = 0, \quad v = 1 \quad \Rightarrow \quad C = -\ln 2$

$$\therefore \quad x = \ln(v+1) - \ln 2$$

Question 11 (d)

Criteria	Marks
• Provides correction solution	3
• Correctly obtains $(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v})$, or equivalent merit OR • Attempts to apply $(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = 0$ using correct $(\underset{\sim}{u} - \underset{\sim}{v})$ and $(\underset{\sim}{u} + \underset{\sim}{v})$	2
• Obtains $(\underset{\sim}{u} - \underset{\sim}{v})$ and $(\underset{\sim}{u} + \underset{\sim}{v})$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{u} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\underset{\sim}{v} = p\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\underset{\sim}{u} - \underset{\sim}{v} = (-2 - p)\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$$

$$\underset{\sim}{u} + \underset{\sim}{v} = (p - 2)\underset{\sim}{i} + 5\underset{\sim}{k}$$

$$(\underset{\sim}{u} - \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = 0$$

$$-(p+2)(p-2) + 5 = 0$$

$$p^2 - 4 = 5$$

$$p = \pm 3$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	4
• Obtains the square root of the discriminant, or equivalent merit	3
• Makes progress towards finding the square root of the discriminant	2
• Correctly obtains the discriminant, or equivalent merit OR • Attempts to use quadratic formula	1

Sample answer:

$$z^2 + 3z + (3 - i) = 0$$

$$\begin{aligned} \Delta &= 9 - 4(3 - i) \\ &= -3 + 4i \end{aligned}$$

Write $-3 + 4i = (x + iy)^2$

$\therefore \quad x^2 - y^2 = -3 \quad$ (real parts)
$ x^2 + y^2 = 5 \quad$ (moduli)

$\therefore \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$
$ \Rightarrow \quad \text{y} = \pm 2$

$\therefore \quad -3 + 4i = (1 + 2i)^2$

Hence, $z = \dfrac{-3 \pm (1 + 2i)}{2}$
$ = -1 + i, -2 - i$

Question 12 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the vertical component of the 200 newton force, or equivalent merit	1

Sample answer:

Resolving vertically,

$$200 \sin 30° + R = mg$$

$$\therefore R = 50 \times 10 - 200 \times \frac{1}{2}$$

$$= 400 \text{ newtons}$$

Question 12 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the horizontal component of the 200 newton force, or equivalent merit	1

Sample answer:

Horizontally,

$$\begin{aligned}\text{Net horizontal force } &= 200 \cos 30° - 0.3R \\ &= 200 \times \frac{\sqrt{3}}{2} - 0.3 \times 400 \\ &\simeq 53.2 \text{ newtons}\end{aligned}$$

Question 12 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Correctly obtains the acceleration, or equivalent merit	1

Sample answer:

From part (b), using $F = ma$,

$50a = 53.2$

$\therefore \dfrac{dV}{dt} = 1.064$

Hence the velocity after 3 seconds is

$$\begin{aligned} V &= \int_0^3 1.064\,dt \\ &= 1.064 \times 3 \\ &= 3.192 \text{ m/s} \end{aligned}$$

Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	3
• Correctly obtains $\underset{\sim}{v}(t) = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$, or equivalent merit OR • Correctly obtains one component of $\underset{\sim}{r}(t)$	2
• Correctly obtains one component of $\underset{\sim}{v}(t)$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} c_1 \\ -gt + c_2 \end{pmatrix}$$

Now at $t = 0$, $\underset{\sim}{v} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix}$

$$\therefore \quad \underset{\sim}{v} = \begin{pmatrix} u\cos\theta \\ u\sin\theta - gt \end{pmatrix}$$

$$\therefore \quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta + c_3 \\ ut\sin\theta - \frac{1}{2}gt^2 + c_4 \end{pmatrix}$$

$$\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_3 = c_4 = 0$$

$$\therefore \quad \underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains a correct Cartesian equation of flight, or equivalent merit	2
• Attempts to eliminate t, or equivalent merit	1

Sample answer:

$$x = ut\cos\theta \quad \Rightarrow \quad t = \frac{x}{u\cos\theta}$$

$$y = ut\sin\theta - \frac{1}{2}gt^2$$

$$= \frac{ux\sin\theta}{u\cos\theta} - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

$$= x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2}\left(1 + \tan^2\theta\right)$$

$$= -\frac{gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right)$$

Question 12 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the discriminant $\Delta = \dfrac{4u^4}{g^2x^2} - 4$, or equivalent merit	1

Sample answer:

From part (ii), we require the quadratic to have two distinct real roots, when $x = R$.

$$\Delta = \frac{4u^4}{g^2R^2} - 4$$

$$= \frac{4\left(u^4 - g^2R^2\right)}{g^2R^2}$$

Given $u^2 > gR \quad \Rightarrow \quad u^4 > g^2R^2$

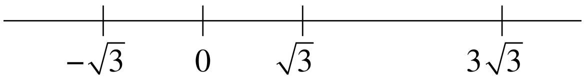
$\therefore \Delta > 0$

Hence there are two distinct values for $\tan\theta$, hence two distinct values for θ, as $\tan\theta$ is increasing.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Correctly finds any two of amplitude, n, function with consistent phase	2
• Correctly finds one of amplitude, n, function with consistent phase	1

Sample answer:



The equation has the form

$$x(t) = A\cos(nt) + \sqrt{3}$$

Now $A = 2\sqrt{3}$ and $\frac{2\pi}{n} = \frac{\pi}{3}$, so $n = 6$

$$\therefore\ x(t) = 2\sqrt{3}\cos(6t) + \sqrt{3}$$

Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly finds either λ_1 or λ_2, or equivalent merit	2
• Finds one equation linking λ_1 and λ_2, or equivalent merit	1

Sample answer:

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

Equating components,

$$\begin{aligned} 3 + \lambda_1 &= 3 - 2\lambda_2 \quad (1) \\ -1 + 2\lambda_1 &= -6 + \lambda_2 \quad (2) \\ 7 + \lambda_1 &= 2 + 3\lambda_2 \quad (3) \end{aligned}$$

(1) $\Rightarrow \lambda_1 = -2\lambda_2$

(2) $\Rightarrow -1 - 4\lambda_2 = -6 + \lambda_2 \Rightarrow \lambda_2 = 1 \Rightarrow \lambda_1 = -2.$

As a check, using equation (3),

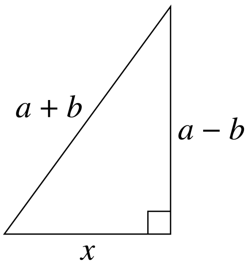
Left hand side = 5, Right hand side = 5

$\therefore$ Point of intersection is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Applies Pythagoras to find x, or equivalent merit OR • Observes that $x \le a + b$	1

Sample answer:



By Pythagoras,

$$(a+b)^2 = x^2 + (a-b)^2$$
$$\therefore x = 2\sqrt{ab}$$

Now the hypotenuse is the longest side so

$$a + b \ge 2\sqrt{ab}$$
$$\therefore \frac{a+b}{2} \ge \sqrt{ab}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Put $a = p^2,\ b = (2q)^2$

Then $\dfrac{p^2 + 4q^2}{2} \ge \sqrt{4p^2q^2}$

So $\quad p^2 + 4q^2 \ge 4pq$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$e^{in\theta}+e^{-in\theta}$$

$$=(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)$$

$$=2\cos(n\theta)$$

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly uses binomial theorem and groups conjugate pairs, or equivalent merit	2
• Uses part (a) to obtain $\left(e^{i\theta}+e^{-i\theta}\right)^4=16\cos^4\theta$ OR • Attempts to use binomial theorem, or equivalent merit	1

Sample answer:

$$\left(e^{i\theta}+e^{-i\theta}\right)^4=e^{4i\theta}+4e^{3i\theta}\cdot e^{-i\theta}+6e^{2i\theta}\cdot e^{-2i\theta}+4e^{i\theta}\cdot e^{-3i\theta}+e^{-4i\theta}$$

$$=\left(e^{4i\theta}+e^{-4i\theta}\right)+4\left(e^{2i\theta}+e^{-2i\theta}\right)+6$$

$$=2\cos4\theta+8\cos2\theta+6$$

Also $\left(e^{i\theta}+e^{-i\theta}\right)^4=2^4\cos^4\theta$

$$\therefore \quad \cos^4\theta=\frac{1}{8}(\cos4\theta+4\cos2\theta+3)$$

Question 13 (d) (iii)

Criteria	Marks
• Provides correct solution	2
• Uses part (a) and attempts to integrate, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) d\theta$$

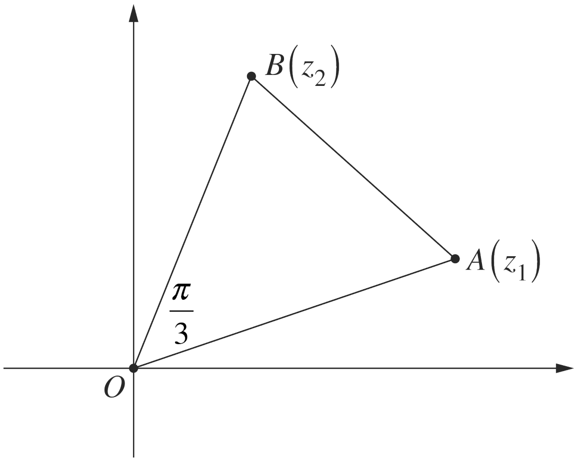
$$= \frac{1}{8}\left(\frac{\sin 4\theta}{4} + 2\sin 2\theta + 3\theta\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left(\frac{3\pi}{2}\right) = \frac{3\pi}{16}$$

Question 14 (a) (i)

Criteria	Marks
• Provides correction solution	2
• Observes either that $\left\lvert z_2 \right\rvert = \left\lvert z_1 \right\rvert$, or that multiplying by $e^{\frac{i\pi}{3}}$ rotates through an angle of $\frac{\pi}{3}$	1

Sample answer:



Multiplying z_1 by $e^{\frac{i\pi}{3}}$ rotates it anticlockwise through an angle $\frac{\pi}{3}$, giving z_2.

Also $\left\lvert z_1 \right\rvert = \left\lvert z_2 \right\rvert$.

So $\triangle OAB$ is isosceles giving all angles equal to $\frac{\pi}{3}$ and hence $\triangle OAB$ is equilateral.

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Factors $z_2^3 + z_1^3$, or equivalent merit OR • Simplifies either side of $z_1^2 + z_2^2 = z_1 z_2$ using $z_2 = e^{\frac{i\pi}{3}} z_1$	2
• Observes that $z_2^3 = e^{i\pi} z_1^3$, or equivalent merit OR • Uses $z_2 = e^{\frac{i\pi}{3}} z_1$ in both sides of $z_1^2 + z_2^2 = z_1 z_2$	1

Sample answer:

$$z_2 = e^{\frac{i\pi}{3}} z_1$$

$$\therefore \quad z_2^3 = e^{i\pi} z_1^3 = -z_1^3$$

$$\therefore \quad z_1^3 + z_2^3 = 0$$

$$\left(z_1 + z_2\right)\left(z_1^2 + z_2^2 - z_1 z_2\right) = 0$$

Now $z_1 \neq -z_2$ so $z_1^2 + z_2^2 = z_1 z_2$.

OR

$$\begin{aligned}
\text{LHS} \quad = \quad z_1^2 + z_2^2 \quad &= \left(e^{\frac{2\pi i}{3}} + 1\right) z_1^2 \\
&= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + 1\right) z_1^2 \\
&= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_1^2
\end{aligned}$$

$$\begin{aligned}
\text{RHS} \quad = \quad z_1 z_2 \quad &= e^{\frac{i\pi}{3}} z_1^2 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) z_1^2 \\
&= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_1^2
\end{aligned}$$

Therefore $z_1^2 + z_2^2 = z_1 z_2$.

Question 14 (b)

Criteria	Marks
• Provides correct solution	4
• Integrates to find v as a function of t, evaluating the constant, or equivalent merit	3
• Correctly applies partial fractions and attempts to integrate, or equivalent merit	2
• Correctly separates variables and attempts to apply partial fractions, or equivalent merit	1

Sample answer:

$$\frac{dv}{dt} = 10\left(1-(kv)^2\right)$$

$$\begin{aligned} t &= \frac{1}{10}\int \frac{1}{(1-kv)(1+kv)}\,dv \\ &= \frac{1}{20}\int \frac{1}{1-kv}+\frac{1}{1+kv}\,dv \qquad \text{(partial fractions)} \\ &= \frac{1}{20}\times\frac{1}{k}\Big[\ln(1+kv)-\ln(1-kv)\Big]+C \\ &= \frac{1}{20k}\ln\left(\frac{1+kv}{1-kv}\right)+C \end{aligned}$$

When $t=0$, $v=0$, so $C=0$.

When $t=5$, we have

$$\ln\left(\frac{1+kv}{1-kv}\right) = 5\times 20\times 0.01 = 1$$

$$\therefore \qquad 1+0.01v = e(1-0.01v)$$

$$v = 100\left(\frac{e-1}{e+1}\right) \simeq 46.2 \text{ m/s}$$

Question 14 (c)

Criteria	Marks
• Provides correct solution	4
• Correctly proves the inductive step OR • Verifies initial case, $n = 2$, AND assumes true for k and attempts to verify true for $k + 1$	3
• Assumes true for k and attempts to verify true for $k + 1$	2
• Verifies initial case, $n = 2$	1

Sample answer:

Let $P(n)$ be the given proposition.

$P(2)$ is true since $\frac{1}{2^2} < \frac{1}{2}$

Let k be an integer for which $P(k)$ is true.

Thus we assume that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < \frac{k-1}{k}$$

Consider $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2}$

$$< \frac{1}{(k+1)^2} + \frac{k-1}{k}$$

$$= \frac{k + (k+1)^2 (k-1)}{(k+1)^2 k}$$

$$< \frac{(k+1) + (k+1)^2 (k-1)}{(k+1)^2 k}$$

$$= \frac{1 + k^2 - 1}{(k+1)k} = \frac{k}{k+1}$$

$\therefore$ $P(k+1)$ is true.

$\therefore$ $P(n)$ true for all integers $n \geq 2$ by induction.

Question 14 (d)

Criteria	Marks
• Provides correct solution	3
• Assumes $\log_n(n+1)$ is rational and attempts to eliminate the logarithm, or equivalent merit	2
• Attempts to use proof by contradiction, or equivalent merit	1

Sample answer:

Suppose $\log_n(n+1)$ is rational.

Then $\log_n(n+1) = \dfrac{p}{q}$, where p, q are integers and $q \neq 0$.

$\therefore\ n+1 = n^{\frac{p}{q}}$

so $(n+1)^q = n^p$

If n is even then RHS is even, LHS is odd – a contradiction.

If n is odd then RHS is odd, LHS is even – a contradiction.

Hence in either case we have a contradiction, so $\log_n(n+1)$ is irrational.

Question 15 (a) (i)

Criteria	Marks
• Provides correct proof	2
• Factors $k^3 + 1$ OR • Writes $k + 1 = 3j$, or equivalent merit	1

Sample answer:

Suppose $k + 1$ is divisible by 3, then $k + 1 = 3j$, for some integer j.

$$\text{Now } k^3+1 = (k+1)(k^2-k+1)$$

$$= 3j(k^2-k+1)$$

Hence $k^3 + 1$ is divisible by 3.

Question 15 (a) (ii)

Criteria	Marks
• Provides correct statement	1

Sample answer:

If $k^3 + 1$ is not divisible by 3, then $(k + 1)$ is not divisible by 3.

Question 15 (a) (iii)

Criteria	Marks
• Provides correct solution	3
• Considers cases to justify their statement, or equivalent merit	2
• Correctly states the converse or equivalent merit	1

Sample answer:

The converse states that:

‘If $k^3 + 1$ is divisible by 3, then $(k + 1)$ is divisible by 3’.

This statement is true.

The integer k must be of the form $3j$, $3j + 1$ or $3j - 1$, for some integer j.

Then $k^3 + 1 = 27j^3 + 1$ or $(3j + 1)^3 + 1$ or $(3j - 1)^3 + 1$.

Only the third case gives $k^3 + 1$ divisible by 3.

OR

Using proof by contradiction, suppose $k^3 + 1$ is divisible by 3.

But $(k + 1)$ is not divisible by 3.

$\therefore$ $k^2 - k + 1$ is divisible by 3

$\therefore$ $(k + 1)^2 - 3k$ is divisible by 3

$\therefore$ $(k + 1)^2$ and hence $(k + 1)$ is divisible by 3.

This is a contradiction.

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Recognises that $\overrightarrow{CB} = \frac{m}{n}\overrightarrow{AC}$, or equivalent merit	1

Sample answer:

$$\overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{OB}$$

$$\therefore\ \underset{\sim}{a} + \overrightarrow{AC} + \frac{m}{n}\overrightarrow{AC} = \overrightarrow{OB}$$

$$\therefore\ \overrightarrow{AC}\left(1+\frac{m}{n}\right) = \underset{\sim}{b} - \underset{\sim}{a}$$

$$\therefore\ \overrightarrow{AC} = \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \underset{\sim}{a} + \frac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right) \\ &= \underset{\sim}{a}\left(1 - \frac{n}{m+n}\right) + \frac{n}{m+n}\left(\underset{\sim}{b}\right) \\ &= \frac{m}{m+n}\underset{\sim}{a} + \frac{n}{m+n}\underset{\sim}{b}\end{aligned}$$

Question 15 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Finds equations of lines OS and PR and attempts to solve for T OR • Writes $\overrightarrow{OT}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{r}$ in two ways and attempts to solve	2
• Finds equation of line OS OR • Writes $\overrightarrow{OT}$ as a multiple of $\overrightarrow{OS} = \underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}$, or equivalent merit	1

Sample answer:

$$\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{SR}$$

$$= \underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}$$

Equation of line OS is $\underset{\sim}{x} = \left(\underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}\right)\lambda$.

Also equation of line PR is $\underset{\sim}{x} = \underset{\sim}{p} + \left(\underset{\sim}{r} - \underset{\sim}{p}\right)\mu$.

Equating, we have

$$\underset{\sim}{p}\left(\frac{1}{2}\lambda\right) + \underset{\sim}{r}\lambda = \underset{\sim}{p}(1-\mu) + \underset{\sim}{r}\mu$$

$$\therefore\ \frac{1}{2}\lambda = 1-\mu \text{ and } \lambda = \mu \Rightarrow \lambda = \frac{2}{3}$$

Hence point of intersection is

$$\overrightarrow{OT} = \frac{2}{3}\left(\underset{\sim}{r} + \frac{1}{2}\underset{\sim}{p}\right) = \frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$$

Question 15 (b) (iv)

Criteria	Marks
• Provides correct solution	1

Sample answer:

By part (ii) the point which divides PR in the ratio $2:1$ is given by $\frac{2}{3}\underset{\sim}{r} + \frac{1}{3}\underset{\sim}{p}$, but this is $\overrightarrow{OT}$.

$\therefore$ T divides the interval PR in the ratio $2:1$.

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to balance forces, or equivalent merit	1

Sample answer:

Let a be the acceleration of the masses. Comparing forces,

$$4gm - kv - 2gm - kv = 2ma + 4ma$$

$$\therefore\ 2gm - 2kv = 6am$$

$$\therefore\ a = \frac{gm - kv}{3m}$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains $t = \frac{3m}{k}\ln\left(\frac{gm}{gm - kv}\right)$, or equivalent merit	2
• Integrates $\frac{dt}{dv}$, or equivalent merit	1

Sample answer:

From part (i), $\frac{dv}{dt} = \frac{gm - kv}{3m}$

$$\therefore t = \int_0^v \frac{3m}{gm - kv}\,dv$$

$$= -\frac{3m}{k}\ln\left(gm - kv\right)\Big|_0^v$$

$$= -\frac{3m}{k}\left[\ln\left(gm - kv\right) - \ln\left(gm\right)\right]$$

$$= \frac{3m}{k}\ln\left(\frac{gm}{gm - kv}\right)$$

When $t = \frac{3m}{k}\ln 2$

We have $\frac{gm}{gm - kv} = 2$

$$\Rightarrow 1 - \frac{kv}{gm} = \frac{1}{2}$$

$$\Rightarrow v = \frac{gm}{2k}$$

Question 16 (b) (i)

Criteria	Marks
• Provides correct proof	3
• Correctly applies integration by parts and attempts to reduce to integrals involving only $\sin(2\theta)$, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin(2\theta)\,\sin^{2n}(2\theta)\,d\theta$$

$$u = \sin^{2n}(2\theta) \qquad \frac{dv}{d\theta} = \sin 2\theta$$

$$\frac{du}{d\theta} = 4n\sin^{2n-1}(2\theta)\cos 2\theta$$

$$v = -\frac{\cos(2\theta)}{2}$$

$$= \frac{1}{2}\sin^{2n}(2\theta) \times -\cos(2\theta)\Big|_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)\,\cos^2(2\theta)\,d\theta$$

$$\therefore\ I_n = 2n\left[\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)\,d\theta - \int_0^{\frac{\pi}{2}} \sin^{2n+1}(2\theta)\,d\theta\right]$$

$$I_n(2n+1) = 2n\,I_{n-1}$$

$$\therefore I_n = \frac{2n}{(2n+1)}I_{n-1}.$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Writes I_n in terms of I_0 and evaluates I_0, or equivalent merit	2
• Calculates I_0 OR • Writes I_n in terms of I_{n-2} OR • Equivalent merit	1

Sample answer:

Applying the reduction formula,

$$I_n = \frac{2n}{2n+1} \times I_{n-1}$$

$$= \frac{2n(2n-2)}{(2n+1)(2n-1)} \times I_{n-2}$$

$$= \frac{2n(2n-2)(2n-4)\ldots\ldots}{(2n+1)(2n-1)(2n-3)\ldots\ldots} \times \frac{2}{3} I_0$$

$$= \frac{[2n(2n-2)\ldots 2][2n(2n-2)\ldots 2] \times I_0}{(2n+1)(2n)(2n-1)(2n-2)\ldots\ldots 3 \times 2}$$

$$= \frac{2^{2n}(n!)^2 \times I_0}{(2n+1)!}$$

Now $I_0 = \displaystyle\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta$

$$= -\frac{1}{2}\cos(2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= 1$$

$$\therefore \ I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Question 16 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Obtains an integral of $\sin^{2n+1} 2\theta$, or equivalent merit	2
• Attempts the substitution $x = \sin^2\theta$, or equivalent merit	1

Sample answer:

$$J_n = \int_0^1 x^n (1-x)^n \, dx$$

Put $x = \sin^2\theta$

$$\frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\therefore \ J_n = \int_0^{\frac{\pi}{2}} 2\sin^{2n+1}(\theta)\cos^{2n+1}(\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n}}\left(2\sin\theta\cos\theta\right)^{2n+1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n}}\sin^{2n+1}(2\theta)\, d\theta$$

$$= \frac{1}{2^{2n}} I_n$$

$$= \frac{(n!)^2}{(2n+1)!}$$

Question 16 (b) (iv)

Criteria	Marks
• Provides correct solution	2
• Finds the maximum value of $x(1-x)$, for $0 \le x \le 1$, or equivalent merit	1

Sample answer:

The quadratic expression $x(1-x)$ has maximum value of $\frac{1}{4}$ when $x = \frac{1}{2}$.

$$\therefore\ J_n = \int_0^1 x^n (1-x)^n \, dx \le \int_0^1 \left(\frac{1}{4}\right)^n dx$$

$$= \frac{1}{2^{2n}}$$

$$\therefore\ \frac{(n!)^2}{(2n+1)!} \le \frac{1}{2^{2n}}$$

$$\Rightarrow \left(2^n n!\right)^2 \le \left(2n+1\right)!$$

2020 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	MEX-V1 Further Work with Vectors	MEX12–3
2	1	MEX-N2 Using Complex Numbers	MEX12–4
3	1	MEX-V1 Further Work with Vectors	MEX12–3
4	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
5	1	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
6	1	MEX-C1 Further Integration	MEX12–5
7	1	MEX-P1 The Nature of Proof	MEX12–2
8	1	MEX-P1 The Nature of Proof	MEX12–2
9	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
10	1	MEX-C1 Further Integration	MEX12–8

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	MEX-N1 Introduction to Complex Numbers	MEX12–4
11 (a) (ii)	2	MEX-N1 Introduction to Complex Numbers	MEX12–4
11 (b)	3	MEX-C1 Further Integration	MEX12–5
11 (c)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
11 (d)	3	MEX-V1 Further Work with Vectors	MEX12–3
11 (e)	4	MEX-N2 Using Complex Numbers	MEX12–4
12 (a) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (a) (ii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (a) (iii)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (i)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (ii)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
12 (b) (iii)	2	MEX-M1 Applications of Calculus to Mechanics 4	MEX12–7
13 (a)	3	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
13 (b)	3	MEX-V1 Further Work with Vectors	MEX12–3
13 (c) (i)	2	MEX-P1 The Nature of Proof	MEX12–2

Question	Marks	Content	Syllabus outcomes
13 (c) (ii)	1	MEX-P1 The Nature of Proof	MEX12–2
13 (d) (i)	1	MEX-N1 Introduction To Complex Numbers	MEX12–4
13 (d) (ii)	3	MEX-N1 Introduction To Complex Numbers	MEX12–4
13 (d) (iii)	2	MEX-N1 Introduction To Complex Numbers	MEX12–5
14 (a) (i)	2	MEX-N2 Using Complex Numbers	MEX12–4
14 (a) (ii)	3	MEX-N2 Using Complex Numbers	MEX12–4
14 (b)	4	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
14 (c)	4	MEX-P2 Further Proof by Mathematical Induction	MEX12–2, MEX12–8
14 (d)	3	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (i)	2	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (ii)	1	MEX-P1 The Nature of Proof	MEX12–2
15 (a) (iii)	3	MEX-P1 The Nature of Proof	MEX12–2
15 (b) (i)	2	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (ii)	1	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (iii)	3	MEX-V1 Further Work with Vectors	MEX12–3
15 (b) (iv)	1	MEX-V1 Further Work with Vectors	MEX12–3
16 (a) (i)	2	MEX-M1 Applications of Calculus to Mechanics	MEX12–6
16 (a) (ii)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (i)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (ii)	3	MEX-C1 Further Integration	MEX12–5
16 (b) (iii)	2	MEX-C1 Further Integration	MEX12–5
16 (b) (iv)	2	MEX-P1 The Nature of Proof	MEX12–2