NSW EDUCATION STANDARDS AUTHORITY

2017 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–18)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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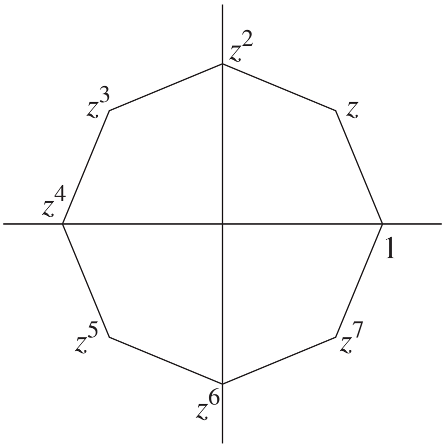
1 The complex number z is chosen so that $1, z, \ldots, z^7$ form the vertices of the regular polygon shown.



Which polynomial equation has all of these complex numbers as roots?

A. $x^7 - 1 = 0$

B. $x^7 + 1 = 0$

C. $x^8 - 1 = 0$

D. $x^8 + 1 = 0$

2 Suppose ℓ is a line and S is a point NOT on ℓ.

The point P moves so that the distance from P to S is half the perpendicular distance from P to ℓ.

Which conic best describes the locus of P?

A. A circle

B. An ellipse

C. A parabola

D. A hyperbola

3 Which complex number lies in the region $2 < |z-1| < 3$?

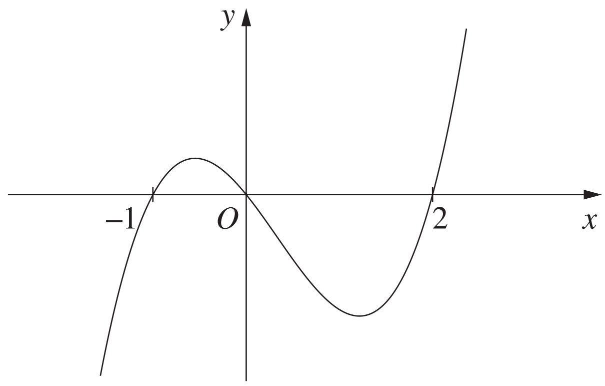
A. $1+\sqrt{3}i$

B. $1+3i$

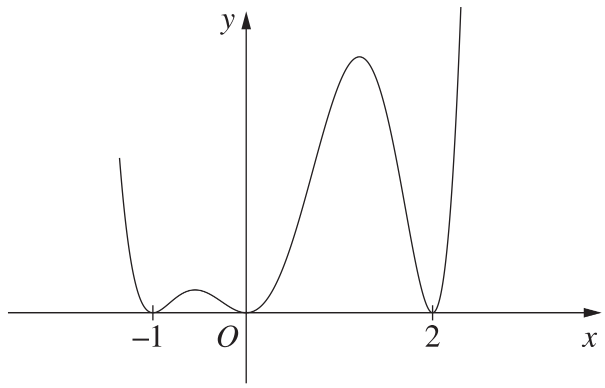
C. $2+i$

D. $3-i$

4 The graph of the function $y = f(x)$ is shown.



A second graph is obtained from the function $y = f(x)$.



Which equation best represents the second graph?

A. $y = \sqrt{f(x)}$

B. $y^2 = f(x)$

C. $y = [f(x)]^2$

D. $y = f(x^2)$

5 The polynomial $p(x) = x^3 - 2x + 2$ has roots α, β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

A. -10

B. -6

C. -2

D. 0

6 It is given that $z = 2 + i$ is a root of $z^3 + az^2 - 7z + 15 = 0$, where a is a real number.

What is the value of a?

A. -1

B. 1

C. 7

D. -7

7 It is given that $f(x)$ is a non-zero even function and $g(x)$ is a non-zero odd function.

Which expression is equal to $\int_{-a}^{a} f(x) + g(x)\,dx$?

A. $2\int_{0}^{a} f(x)\,dx$

B. $2\int_{0}^{a} g(x)\,dx$

C. $\int_{-a}^{a} g(x)\,dx$

D. $2\int_{0}^{a} f(x) + g(x)\,dx$

8 Suppose that $f(x)$ is a non-zero odd function.

Which of the functions below is also odd?

A. $f\left(x^2\right)\cos x$

B. $f(f(x))$

C. $f\left(x^3\right)\sin x$

D. $f\left(x^2\right) - f(x)$

9 A particle is travelling on the circle with equation $x^2 + y^2 = 16$.

It is given that $\frac{dx}{dt} = y$.

Which statement about the motion of the particle is true?

A. $\frac{dy}{dt} = x$ and the particle travels clockwise

B. $\frac{dy}{dt} = x$ and the particle travels anticlockwise

C. $\frac{dy}{dt} = -x$ and the particle travels clockwise

D. $\frac{dy}{dt} = -x$ and the particle travels anticlockwise

10 Suppose $f(x)$ is a differentiable function such that

$$\frac{f(a)+f(b)}{2} \geq f\left(\frac{a+b}{2}\right), \text{ for all } a \text{ and } b.$$

Which statement is always true?

A. $\displaystyle\int_0^1 f(x)\,dx \geq \frac{f(0)+f(1)}{2}$

B. $\displaystyle\int_0^1 f(x)\,dx \leq \frac{f(0)+f(1)}{2}$

C. $f'\left(\frac{1}{2}\right) \geq 0$

D. $f'\left(\frac{1}{2}\right) \leq 0$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

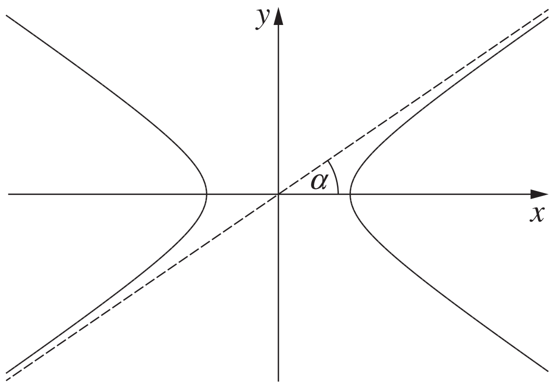
Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 1 - \sqrt{3}i$ and $w = 1 + i$.

(i) Find the exact value of the argument of z. **1**

(ii) Find the exact value of the argument of $\frac{z}{w}$. **2**

(b) An asymptote to the hyperbola $\frac{x^2}{12} - \frac{y^2}{4} = 1$ makes an angle α with the positive x-axis, as shown. **2**



Find the value of α.

(c) Sketch the region in the Argand diagram where **2**

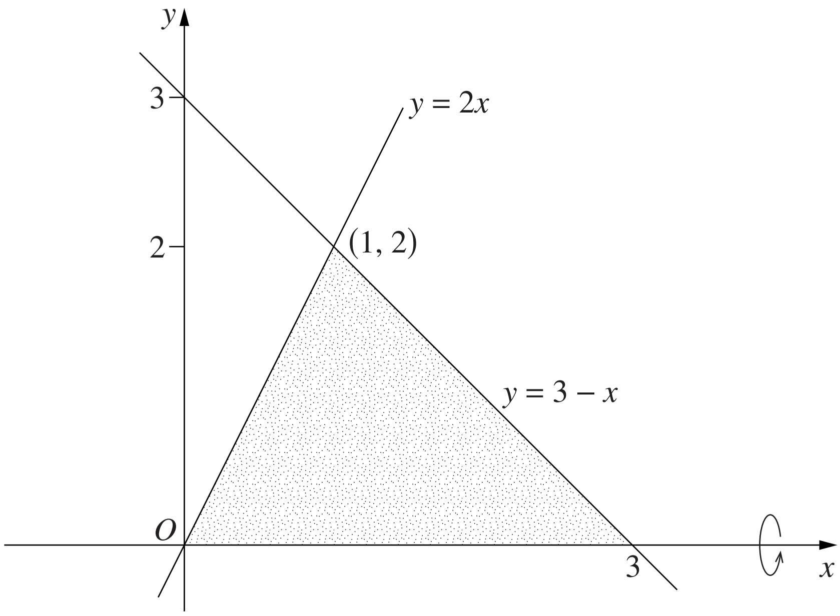
$$-\frac{\pi}{4} \le \arg z \le 0 \quad \text{and} \quad |z - 1 + i| \le 1.$$

Question 11 continues on page 8

(d) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, evaluate **3**

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}\,d\theta.$$

(e) The region bounded by the lines $y = 3 - x$, $y = 2x$ and the x-axis is rotated about the x-axis. **2**



Use the method of cylindrical shells to find an integral whose value is the volume of the solid of revolution formed. Do NOT evaluate the integral.

(f) Using the substitution $x = \sin^2\theta$, or otherwise, evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x) = \dfrac{e^x - 1}{e^x + 1}$.

(i) Show that $f(x)$ is increasing for all x. **1**

(ii) Show that $f(x)$ is an odd function. **1**

(iii) Describe the behaviour of $f(x)$ for large positive values of x. **1**

(iv) Hence sketch the graph of $f(x) = \dfrac{e^x - 1}{e^x + 1}$. **1**

(v) Hence, or otherwise, sketch the graph of $y = \dfrac{1}{f(x)}$. **1**

(b) Solve the quadratic equation $z^2 + (2 + 3i)z + (1 + 3i) = 0$, giving your answers in the form $a + bi$, where a and b are real numbers. **3**

(c) Find $\displaystyle\int x \tan^{-1} x \; dx$. **3**

(d) Let $P(x)$ be a polynomial.

(i) Given that $(x - \alpha)^2$ is a factor of $P(x)$, show that **2**

$$P(\alpha) = P'(\alpha) = 0.$$

(ii) Given that the polynomial $P(x) = x^4 - 3x^3 + x^2 + 4$ has a factor $(x - \alpha)^2$, find the value of α. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $\dfrac{r+s}{2} \geq \sqrt{rs}$ for $r \geq 0$ and $s \geq 0$. **1**

(b) Let a, b and c be real numbers. Suppose that $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ has roots α, $\dfrac{1}{\alpha}$, β, $\dfrac{1}{\beta}$, where $\alpha > 0$ and $\beta > 0$.

(i) Prove that $a = c$. **2**

(ii) Using the inequality in part (a), show that $b \geq 6$. **2**

(c) A particle is projected upwards from ground level with initial velocity $\dfrac{1}{2}\sqrt{\dfrac{g}{k}}$ m s^{-1}, where g is the acceleration due to gravity and k is a positive constant. The particle moves through the air with speed v m s^{-1} and experiences a resistive force. **4**

The acceleration of the particle is given by $\ddot{x} = -g - kv^2$ m s^{-2}. Do NOT prove this.

The particle reaches a maximum height, H, before returning to the ground.

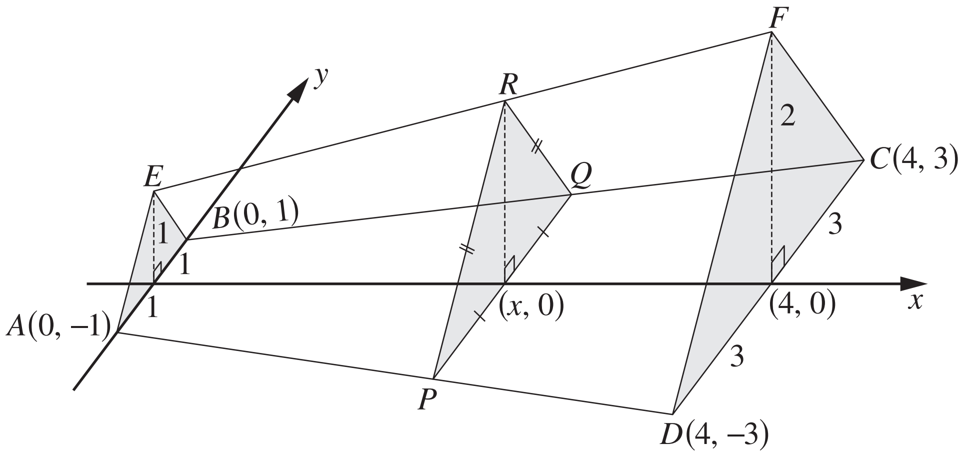
Using $\ddot{x} = v\dfrac{dv}{dx}$, or otherwise, show that $H = \dfrac{1}{2k}\log_e\left(\dfrac{5}{4}\right)$ metres.

Question 13 continues on page 11

Question 13 (continued)

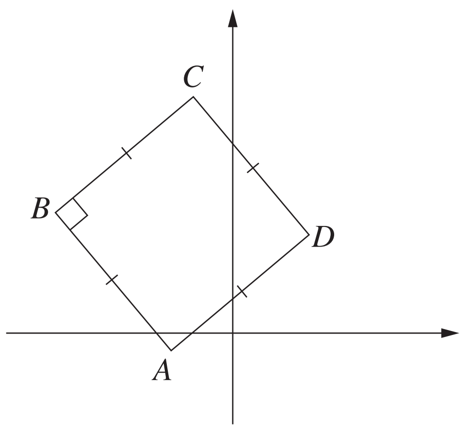
(d) The trapezium whose vertices are $A(0, -1)$, $B(0, 1)$, $C(4, 3)$ and $D(4, -3)$ forms the base of a solid. **4**

Each cross-section perpendicular to the x-axis is an isosceles triangle. The height of the isosceles triangle ABE with base AB, is 1. The height of the isosceles triangle DCF with base DC, is 2. The cross-section through the point $(x, 0)$ is the isosceles triangle PQR, where R lies on the line EF, as shown in the diagram.



Find the volume of the solid.

(e) The points A, B, C and D on the Argand diagram represent the complex numbers a, b, c and d respectively. The points form a square as shown on the diagram. **2**



By using vectors, or otherwise, show that $c = (1 + i)d - ia$.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) It is given that

$$x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2).$$

(i) Find A and B so that **1**

$$\frac{16}{x^4 + 4} = \frac{A + 2x}{x^2 + 2x + 2} + \frac{B - 2x}{x^2 - 2x + 2}.$$

(ii) Hence, or otherwise, show that for any real number m, **2**

$$\int_0^m \frac{16}{x^4 + 4}\,dx = \ln\left(\frac{m^2 + 2m + 2}{m^2 - 2m + 2}\right) + 2\tan^{-1}(m + 1) + 2\tan^{-1}(m - 1).$$

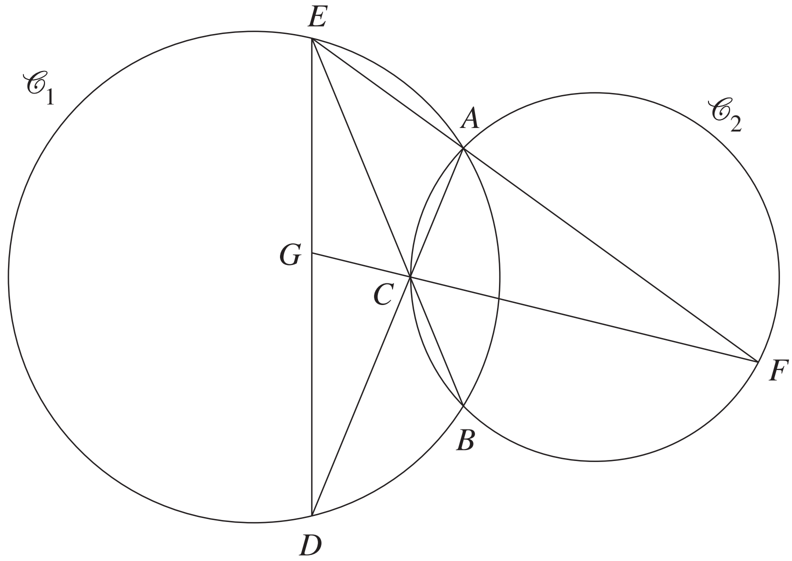
(iii) Find the limiting value as $m \longrightarrow \infty$ of **1**

$$\int_0^m \frac{16}{x^4 + 4}\,dx.$$

Question 14 continues on page 13

(b) Two circles, $\mathscr{C}_1$ and $\mathscr{C}_2$, intersect at the points A and B. Point C is chosen on the arc AB of $\mathscr{C}_2$ as shown in the diagram.

The line segment AC produced meets $\mathscr{C}_1$ at D.
The line segment BC produced meets $\mathscr{C}_1$ at E.
The line segment EA produced meets $\mathscr{C}_2$ at F.
The line segment FC produced meets the line segment ED at G.



Copy or trace the diagram into your writing booklet.

(i) State why $\angle EAD = \angle EBD$. **1**

(ii) Show that $\angle EDA = \angle AFC$. **1**

(iii) Hence, or otherwise, show that B, C, G and D are concyclic points. **3**

Question 14 continues on page 14

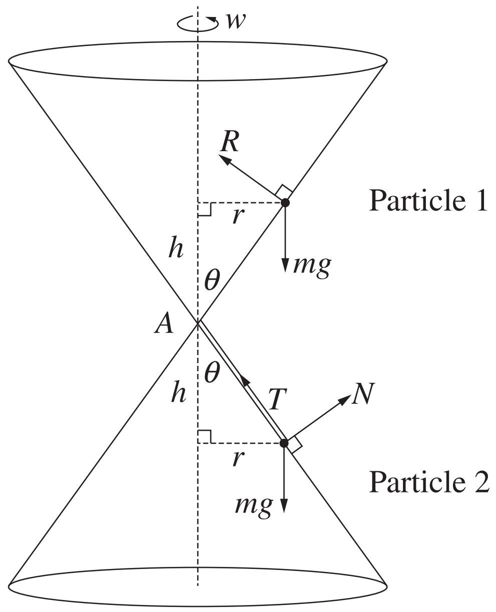
(c) A smooth double cone with semi-vertical angle $\theta < \frac{\pi}{2}$ is rotating about its axis with constant angular velocity w.

Two particles, each of mass m, are sitting on the cone as it rotates, as shown in the diagram.

Particle 1 is inside the cone at vertical distance h above the apex, A, and moves in a horizontal circle of radius r.

Particle 2 is attached to the apex A by a light inextensible string so that it sits on the cone at vertical distance h below the apex. Particle 2 also moves in a horizontal circle of radius r.

The acceleration due to gravity is g.



(i) The normal reaction force on Particle 1 is R. **2**

By resolving R into vertical and horizontal components, or otherwise, show that $w^2 = \frac{gh}{r^2}$.

(ii) The normal reaction force on Particle 2 is N and the tension in the string is T. **2**

By considering horizontal and vertical forces, or otherwise, show that

$$N = mg\left(\sin\theta - \frac{h}{r}\cos\theta\right).$$

(iii) Show that $\theta \geq \frac{\pi}{4}$. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^1 x^n \sqrt{1 - x^2}\, dx$, for $n = 0, 1, 2, \ldots$.

(i) Find the value of I_1. **1**

(ii) Using integration by parts, or otherwise, show that for $n \geq 2$ **3**

$$I_n = \left(\frac{n-1}{n+2}\right) I_{n-2}.$$

(iii) Find the value of I_5. **1**

(b) Consider the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, for $x \geq 0$ and $y \geq 0$, where a is a positive constant.

(i) Show that the equation of the tangent to the curve at the point $P(c, d)$ is given by $y\sqrt{c} + x\sqrt{d} = d\sqrt{c} + c\sqrt{d}$. **2**

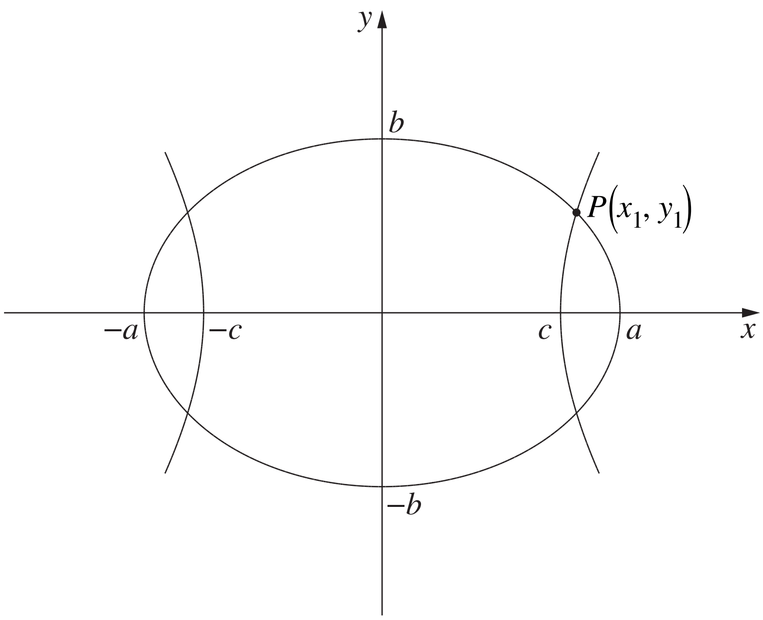
(ii) The tangent to the curve at the point P meets the x and y axes at A and B respectively. Show that $OA + OB = a$, where O is the origin. **3**

Question 15 continues on page 16

(c) The ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, has eccentricity e.

The hyperbola with equation $\frac{x^2}{c^2} - \frac{y^2}{d^2} = 1$, has eccentricity E.

The value of c is chosen so that the hyperbola and the ellipse meet at $P(x_1, y_1)$, as shown in the diagram.



(i) Show that $\frac{x_1{}^2}{y_1{}^2} = \frac{a^2c^2}{(a^2 - c^2)} \times \frac{(b^2 + d^2)}{b^2d^2}$. **2**

(ii) If the two conics have the same foci, show that their tangents at P are perpendicular. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $\alpha = \cos\theta + i\sin\theta$, where $0 < \theta < 2\pi$.

(i) Show that $\alpha^k + \alpha^{-k} = 2\cos k\theta$, for any integer k. **1**

Let $C = \alpha^{-n} + \cdots + \alpha^{-1} + 1 + \alpha + \cdots + \alpha^n$, where n is a positive integer.

(ii) By summing the series, prove that $C = \dfrac{\alpha^n + \alpha^{-n} - \left(\alpha^{n+1} + \alpha^{-(n+1)}\right)}{(1-\alpha)(1-\overline{\alpha})}$. **3**

(iii) Deduce, from parts (i) and (ii), that **2**

$$1 + 2(\cos\theta + \cos 2\theta + \cdots + \cos n\theta) = \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}.$$

(iv) Show that $\cos\dfrac{\pi}{n} + \cos\dfrac{2\pi}{n} + \cdots + \cos\dfrac{n\pi}{n}$ is independent of n. **1**

(b) The hyperbola with equation **2**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has eccentricity 2.

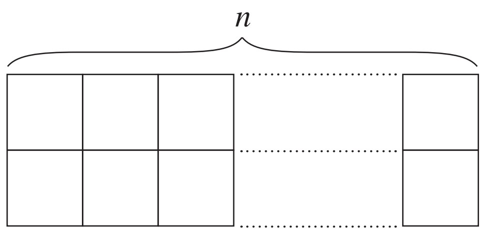
The distance from one of the foci to one of the vertices is 1.

What are the possible values of a?

Question 16 continues on page 18

Question 16 (continued)

(c) A 2 by n grid is made up of two rows of n square tiles, as shown.



The tiles of the 2 by n grid are to be painted so that tiles sharing an edge are painted using different colours. There are x different colours available, where $x \geq 2$.

It is NOT necessary to use all the colours.

Consider the case of the 2 by 2 grid with tiles labelled A, B, C and D, as shown.

A	C
B	D

There are $x(x-1)$ ways to choose colours for the first column containing tiles A and B. Do NOT prove this.

(i) Assume the colours for tiles A and B have been chosen. There are two cases to consider when choosing colours for the second column. Either tile C is the same colour as tile B, or tile C is a different colour from tile B. **2**

By considering these two cases, show that the number of ways of choosing colours for the second column is

$$x^2 - 3x + 3.$$

(ii) Prove by mathematical induction that the number of ways in which the 2 by n grid can be painted is $x(x-1)\left(x^2 - 3x + 3\right)^{n-1}$, for $n \geq 1$. **2**

(iii) In how many ways can a 2 by 5 grid be painted if 3 colours are available and each colour must now be used at least once? **2**

End of paper

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REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180°$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

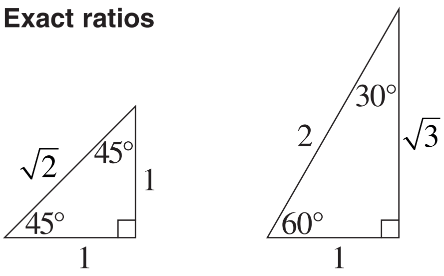
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point–gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \ \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at, at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1, y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0, y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos\left(nt + \alpha\right)$$

$$\ddot{x} = -n^2\left(x - b\right)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

Binomial theorem

$$\left(a + b\right)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$