NSW EDUCATION STANDARDS AUTHORITY

2017 HSC Mathematics Extension 2
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	B
3	D
4	C
5	B
6	A
7	A
8	B
9	C
10	B

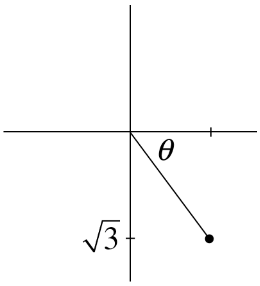
Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct value	1

Sample answer:

$z = 1 - i\sqrt{3}$



$$\arg z = -\frac{\pi}{3}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds argument of w, or equivalent merit	1

Sample answer:

$$w = 1 + i$$

$$\arg w = \frac{\pi}{4}$$

$$\begin{aligned}\arg\left(\frac{z}{w}\right) &= \arg z - \arg w \\ &= \left(-\frac{\pi}{3}\right) - \left(\frac{\pi}{4}\right) \\ &= \frac{-4\pi - 3\pi}{12} \\ &= \frac{-7\pi}{12}\end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	2
• Finds the slope of the asymptote, or equivalent merit	1

Sample answer:

$$\frac{x^2}{12} - \frac{y^2}{4} = 1$$

$$a^2 = 12 \qquad \therefore a = 2\sqrt{3}$$

$$b^2 = 4 \qquad \therefore b = 2$$

$$\tan\alpha = \frac{b}{a} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

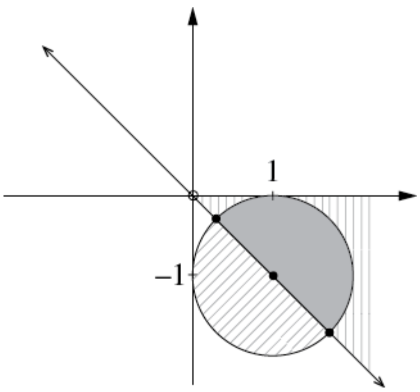
$$\alpha = \frac{\pi}{6}$$

Question 11 (c)

Criteria	Marks
• Provides correct sketch	2
• Sketches one correct region, or equivalent merit	1

Sample answer:

$$-\frac{\pi}{4} \le \arg z \le 0 \quad \text{and} \quad \left|z-(1-i)\right| \le 1$$



Question 11 (d)

Criteria	Marks
• Provides correct solution	3
• Obtains correct primitive, or equivalent merit	2
• Correctly deals with $d\theta$, or equivalent merit	1

Sample answer:

$$t = \tan\frac{\theta}{2}$$

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta} \times d\theta = \int_0^{\sqrt{3}} \frac{\frac{2dt}{1+t^2}}{\frac{1+t^2}{1+t^2} + \frac{1-t^2}{1+t^2}}$$

$$= \int_0^{\sqrt{3}} \frac{2dt}{2}$$

$$= \int_0^{\sqrt{3}} 1 \times dt$$

$$= t\Big]_0^{\sqrt{3}}$$

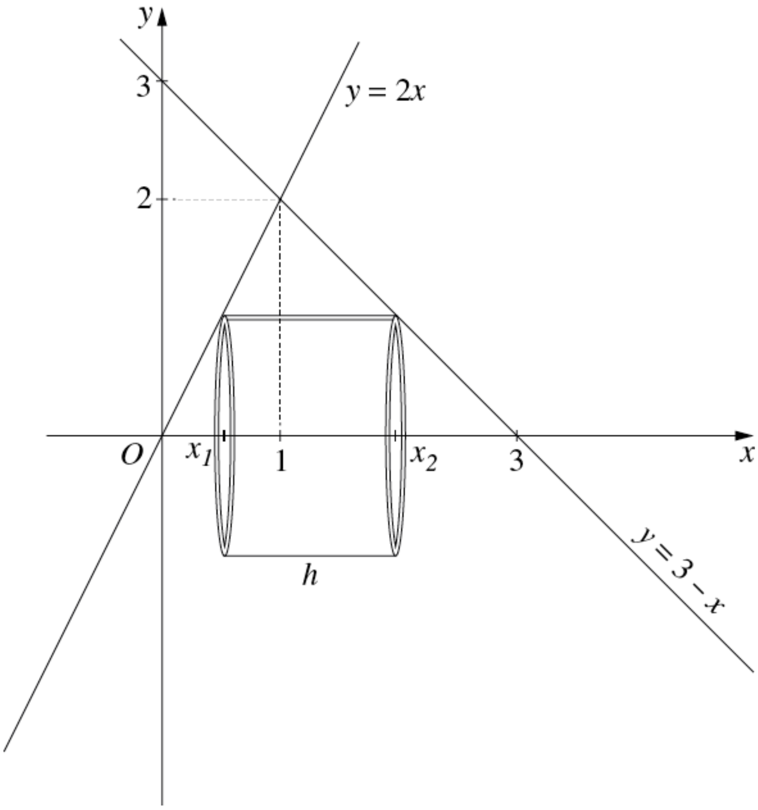
$$= \sqrt{3} - 0$$

$$= \sqrt{3}$$

Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Finds the height of the shells in terms of *y*, or equivalent merit	1

Sample answer:



$$r = y$$

$$\begin{aligned} h &= x_2 - x_1 \\ &= (3 - y) - \left(\frac{y}{2}\right) \\ &= 3 - \frac{3}{2}y \end{aligned}$$

$$V = 2\pi \int_0^2 rh \; dy$$

$$= 2\pi \int_0^2 y \times \left(3 - \frac{3}{2}y\right) \times dy$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Attempts to use a double angle result, or equivalent merit	2
• Obtains correct integrand in terms of θ, or equivalent merit	1

Sample answer:

$$x = \sin^2\theta$$

$$dx = 2\sin\theta\cos\theta\, d\theta$$

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}} \times dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\, d\theta$$

$$= \left.\theta - \frac{1}{2}\sin 2\theta \right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - (0)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

Question 12 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$f(x) = \frac{e^x - 1}{e^x + 1}$$

$$u = e^x - 1$$
$$u' = e^x$$
$$v = e^x + 1$$
$$v' = e^x$$

$$f'(x) = \frac{\left(e^x + 1\right)e^x - \left(e^x - 1\right)e^x}{\left(e^x + 1\right)^2}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{\left(e^x + 1\right)^2}$$

$$= \frac{2e^x}{\left(e^x + 1\right)^2}$$

$f'(x) > 0$ for all x as $2e^x > 0$ and $\left(e^x + 1\right)^2 > 0$

$\therefore$ $f(x)$ is increasing for all x.

Question 12 (a) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$f(-x) = \frac{e^{-x}-1}{e^{-x}+1}$$

$$= \frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1}$$

$$= \frac{\frac{1-e^x}{e^x}}{\frac{1+e^x}{e^x}}$$

$$= \frac{1-e^x}{1+e^x}$$

$$= \frac{-\left(-1+e^x\right)}{e^x+1}$$

$$= -\frac{\left(e^x-1\right)}{e^x+1}$$

$$= -\left[f(x)\right]$$

$\therefore f(-x) = -f(x)$

the function is odd.

Question 12 (a) (iii)

Criteria	Marks
• Provides correct description	1

Sample answer:

$$\lim_{x\to\infty} \frac{e^x-1}{e^x+1} = 1$$

ie $x \to \infty$, $f(x) \to 1$ (from below)

Question 12 (a) (iv)

Criteria	Marks
• Provides correct sketch	1

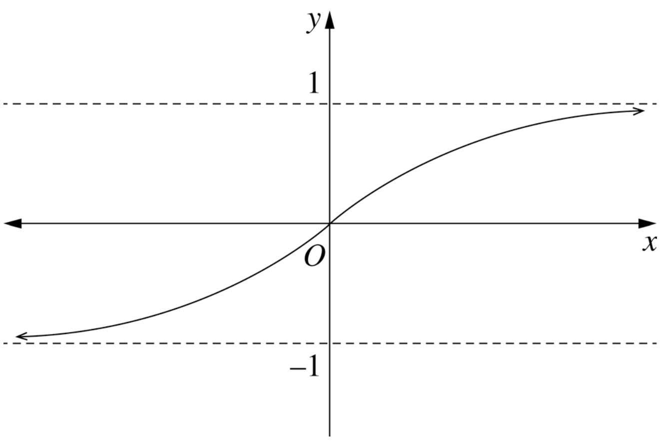
Sample answer:

$$f(0) = \frac{e^0 - 1}{e^0 + 1}$$

$$= \frac{1 - 1}{1 + 1}$$

$$= 0$$

$\therefore$ (0, 0) lies on the curve.



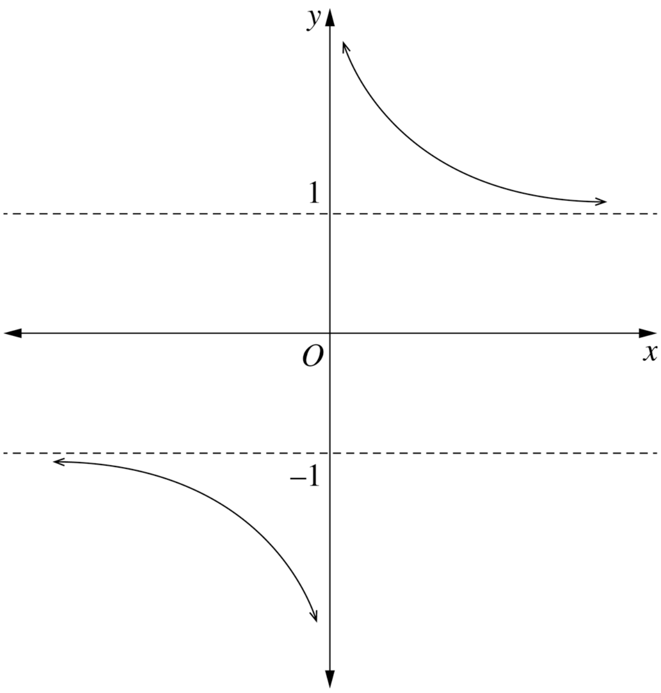
Question 12 (a) (v)

Criteria	Marks
• Provides correct sketch	1

Sample answer:

$$y = \frac{e^x + 1}{e^x - 1}$$

$$y = \frac{1}{f(x)}$$



Question 12 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly evaluates the discriminant, or equivalent merit	2
• Uses quadratic formula, or equivalent merit	1

Sample answer:

$z^2 + (2+3i)z + (1+3i) = 0$

$a = 1$

$b = 2+3i$

$c = 1+3i$

$$z = \frac{-(2+3i) \pm \sqrt{(2+3i)^2 - 4(1)(1+3i)}}{2(1)}$$

$$= \frac{-(2+3i) \pm \sqrt{4+12i+9i^2-4-12i}}{2}$$

$$= \frac{-(2+3i) \pm \sqrt{9i^2}}{2}$$

$$= \frac{-(2+3i) \pm 3i}{2}$$

$z = \dfrac{-2-3i+3i}{2}$ and $z = \dfrac{-2-3i-3i}{2}$

$z = \dfrac{-2}{2}$ and $z = \dfrac{-2-6i}{2}$

$z = -1$ $\quad$ $z = -1-3i$

Question 12 (c)

Criteria	Marks
• Provides correct primitive	3
• Correctly uses parts, or equivalent merit	2
• Attempts to use parts, or equivalent merit	1

Sample answer:

$$I = \int x \tan^{-1} x \; dx$$

Using integration by parts,

$$u = \tan^{-1} x \qquad dv = x dx$$

$$du = \frac{1}{1+x^2} dx \qquad v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2} \left(\frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int 1 - \frac{1}{1+x^2} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \left[x - \tan^{-1} x \right] + c$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2} + c$$

$$= \frac{1}{2} \left(x^2 \tan^{-1} x - x + \tan^{-1} x \right) + c$$

Question 12 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Recognises that $P(x) = (x-\alpha)^2 q(x)$, or equivalent merit	1

Sample answer:

Let $P(x) = (x-\alpha)^2 \times Q(x)$

$$P'(x) = 2(x-\alpha)' \times Q(x) + (x-\alpha)^2 \times Q'(x)$$

$$= (x-\alpha)[2 \times Q(x) + (x-\alpha) \times Q'(x)]$$

$$P(\alpha) = (\alpha-\alpha)^2 \times Q(\alpha)$$

$$= 0$$

$$P'(\alpha) = (\alpha-\alpha)[2 \times Q(\alpha) + (\alpha-\alpha) \times Q'(\alpha)]$$

$$= 0$$

Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Attempt to solve $P'(x) = 0$, or equivalent merit	1

Sample answer:

$$P(x) = x^4 - 3x^3 + x^2 + 4$$

$$P'(x) = 4x^3 - 9x^2 + 2x$$

$$= x\left(4x^2 - 9x + 2\right)$$

$$= x(4x-1)(x-2)$$

Roots of $P'(x)$ are $x = 0, \frac{1}{4}, 2$

$$P(0) = 4$$

$$P\left(\frac{1}{4}\right) = \frac{1029}{256}$$

$$P(2) = 0$$

$\therefore (x-2)^2$ is a factor of $P(x)$

$\alpha = 2$

Question 13 (a)

Criteria	Marks
• Provides correct solution	1

Sample answer:

As $\left(\sqrt{r}-\sqrt{s}\right)^2 \geq 0$

$$r-2\sqrt{rs}+s \geq 0$$

$$\therefore \frac{r+s}{2} \geq \sqrt{rs}$$

Question 13 (b) (i)

Criteria	Marks
• Provides correct proof	2
• Attempts to use sum of the threewise products of roots, or equivalent merit	1

Sample answer:

$-a$ = sum of roots $= \alpha+\dfrac{1}{\alpha}+\beta+\dfrac{1}{\beta}$

$-c$ = sum of products of 3 roots

$$= \alpha\frac{1}{\alpha}\beta+\alpha\frac{1}{\alpha}\frac{1}{\beta}+\alpha\beta\frac{1}{\beta}+\frac{1}{\alpha}\beta\frac{1}{\beta}$$

$$= \beta+\frac{1}{\beta}+\alpha+\frac{1}{\alpha}$$

$$= -a$$

$$\therefore a = c$$

OR

The equation whose roots are reciprocals of the roots is found by replacing x by $\dfrac{1}{x}$.

$$\frac{1}{x^4}+a\frac{1}{x^3}+b\frac{1}{x^2}+c\frac{1}{x}+d$$

ie $1+ax+bx^2+cx^3+dx^4$

Since the reciprocals give the same set of roots we can match the coefficients so that $d = 1$ and $c = a$.

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Uses sum of the roots in pairs, or equivalent merit	1

Sample answer:

b = sum of pairs of roots

$$= \alpha\frac{1}{\alpha} + \alpha\beta + \alpha\frac{1}{\beta} + \beta\frac{1}{\alpha} + \frac{1}{\alpha}\frac{1}{\beta} + \beta\frac{1}{\beta}$$

$$= 1 + \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + 1$$

$$= 1 + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \alpha\beta + \frac{1}{\alpha}\beta$$

Now $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} \geq \frac{2\sqrt{\alpha^2\beta^2}}{\alpha\beta}$ from 13 part (a)

$\therefore \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \geq 2$ and similarly $\alpha\beta + \frac{1}{\alpha\beta} \geq 2$

$\therefore b \geq 2 + 2 + 2 = 6$

Question 13 (c)

Criteria	Marks
• Provides correct solution	4
• Obtains correct expression for x in terms of v, k and g, or equivalent merit	3
• Separates the variables, or equivalent merit	2
• Rewrites acceleration as $v\dfrac{dv}{dx}$, or equivalent merit	1

Sample answer:

$$\ddot{x} = v\frac{dv}{dx} = -g - kv^2$$

$$\therefore \frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$\therefore x = -\int \frac{v}{g + kv^2}\,dv$$

$$= -\frac{1}{2k}\log_e\left(g + kv^2\right) + c$$

when $x = 0$ $\quad v = \dfrac{1}{2}\sqrt{\dfrac{g}{k}}$ so that $v^2 = \dfrac{g}{4k}$

hence $\quad 0 = -\dfrac{1}{2k}\log_e\left(g + \dfrac{kg}{4k}\right) + c$

$$\therefore c = \frac{1}{2k}\log_e\left(\frac{5g}{4}\right)$$

$$\therefore x = \frac{1}{2k}\log_e\left(\frac{5g}{4}\right) - \frac{1}{2k}\log_e\left(g + kv^2\right)$$

Maximum height occurs when $v = 0$

$$\therefore H = \frac{1}{2k}\left[\log_e\left(\frac{5g}{4}\right) - \log_e(g)\right]$$

$$= \frac{1}{2k}\log_e\left(\frac{5}{4}\right)$$

Question 13 (d)

Criteria	Marks
• Provides correct solution	4
• Obtains an integral for the volume, or equivalent merit	3
• Finds the area of the cross-section in terms of x, or equivalent merit	2
• Finds the height of $\triangle PQR$ in terms of x, or equivalent merit	1

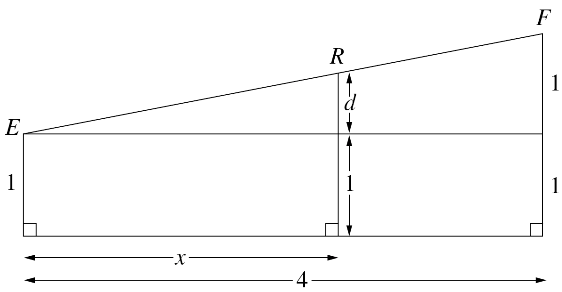
Sample answer:

The line BC has equation $y = \left(\frac{3-1}{4-0}\right)x + 1$

$$y = \frac{1}{2}x + 1$$

$$\therefore QP = 2\left(\frac{1}{2}x + 1\right) = x + 2$$

To find height of $\triangle QRP$



$$\therefore\ d = \frac{x}{4}$$

$$\therefore \text{height} = 1 + d = \frac{x}{4} + 1$$

By similar triangles $\frac{d}{x} = \frac{1}{4}$

$$\therefore \text{ Area } \triangle QRP = \frac{1}{2}QP\left(\frac{x}{4}+1\right)$$

$$= \left(\frac{x}{2}+1\right)\left(\frac{x}{4}+1\right)$$

$$= \frac{x^2}{8}+\frac{3x}{4}+1$$

$$\text{Volume } = \frac{1}{8}\int_0^4 x^2+6x+8\,dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3}+3x^2+8x\right]_0^4$$

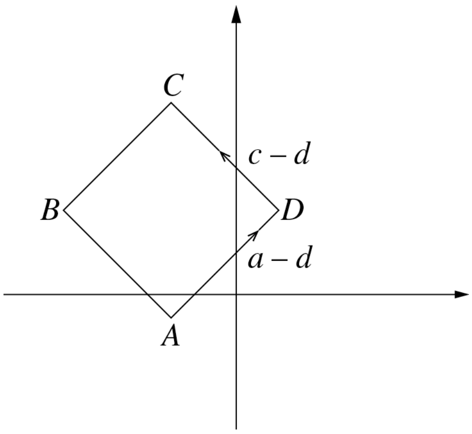
$$= \frac{1}{8}\left[\frac{64}{3}+48+32-0\right]$$

$$= \frac{38}{3} \text{ units}^3$$

Question 13 (e)

Criteria	Marks
• Provides correct solution	2
• Obtains $c - d = i(d - a)$, or equivalent merit	1

Sample answer:



$c - d = i(d - a)$

(rotation by $\dfrac{\pi}{2}$)

$c = d + id - ia$

$= (1 + i)d - ia$

Question 14 (a) (i)

Criteria	Marks
• Provides correct values	1

Sample answer:

Multiply both sides by $x^4 + 4$

$$16 = \left(A + 2x\right)\left(x^2 - 2x + 2\right) + \left(B - 2x\right)\left(x^2 + 2x + 2\right)$$

at $x = 0$ $\quad 16 = 2A + 2B \quad A + B = 8$ —— ①

at $x = 1$ $\quad 16 = (A + 2) + (B - 2)5$

$16 = A + 2 + 5B - 10 \quad A + 5B = 24$ —— ②

Subtract ② − ①

$4B = 16$

$B = 4$

$A = 4$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains correct primitive, or equivalent merit	1

Sample answer:

$$\int_0^m \frac{16}{x^4+4}\,dx$$

$$= \int_0^m \frac{4+2x}{x^2+2x+2} + \int_0^m \frac{4-2x}{x^2-2x+2}\,dx$$

$$= \int_0^m \frac{2}{x^2+2x+2} + \frac{2x+2}{x^2+2x+2}\,dx + \int_0^m \frac{2}{x^2-2x+2} - \frac{2x-2}{x^2-2x+2}\,dx$$

$$= \int_0^m \frac{2}{1+(x+1)^2}\,dx + \Big[\ln\left(x^2+2x+2\right)\Big]_0^m + \int_0^m \frac{2}{1+(x-1)^2}\,dx - \Big[\ln\left(x^2-2x+2\right)\Big]_0^m$$

$$= \Big[2\tan^{-1}(x+1)\Big]_0^m + \ln\left(m^2+2m+2\right) - \log 2 + \Big[2\tan^{-1}(x-1)\Big]_0^m - \ln\left(m^2-2m+2\right) + \log 2$$

$$= 2\tan^{-1}(m+1) - \frac{\pi}{2} + \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m-1) + \frac{\pi}{2}$$

$$= \ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right) + 2\tan^{-1}(m+1) + 2\tan^{-1}(m-1)$$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\int_0^m \frac{16}{x^4+4}\,dx = \ln\left(\frac{1+\frac{2}{m}+\frac{2}{m^2}}{1-\frac{2}{m}+\frac{2}{m^2}}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1)$$

as $m \to \infty$

$$\to \ln(1)+2\left(\frac{\pi}{2}\right)+2\left(\frac{\pi}{2}\right)=2\pi$$

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Construct DB

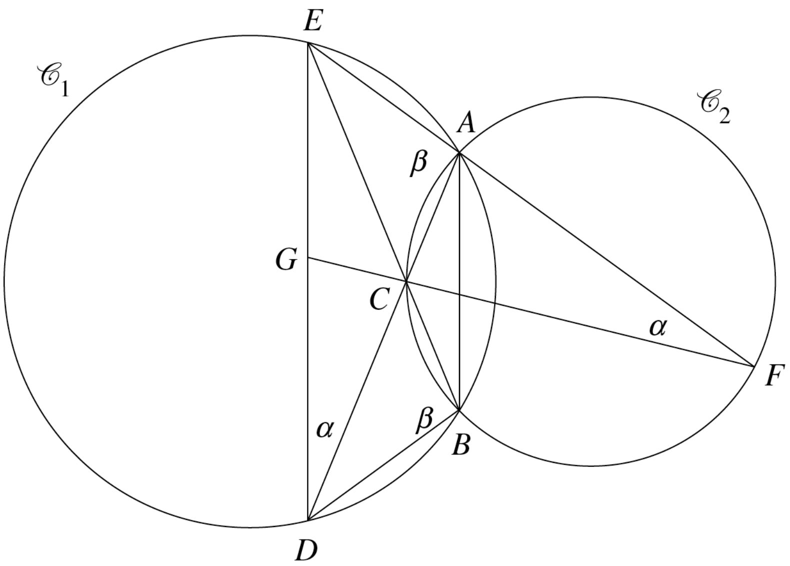
$\angle EAD = \angle EBD$, as they stand on the same arc, ED of C_1.

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Construct AB



$\angle EDA = \angle EBA$ angles standing on arc EA of C_1.

$\angle EBA = \angle CBA$
$= \angle CFA$ angles standing on arc AC in other circle C_2 .

so $\angle EDA = \angle EBA = \angle CFA = \angle AFC$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Makes substantial progress, or equivalent merit	2
• Attempts to find an exterior angle, or equivalent merit	1

Sample answer:

Let $\angle AFC = \angle EDA = \alpha$
$\angle EAD = \angle EBD = \beta$

In $\triangle EAD$, $\angle AED = \pi - \alpha - \beta$

In $\triangle EFG$, $\angle EGF = \pi - (\alpha + (\pi - \alpha - \beta))$
$= \beta$

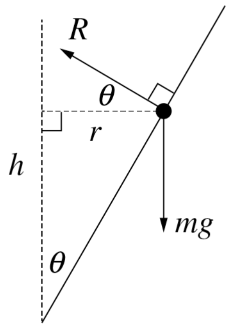
so $\angle EGF = \angle EBD = \beta$

Exterior angle (*EGF)* equals opposite interior angle (*CBD*) and so *BCGD* is a cyclic quadrilateral.

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains one correct component of R, or equivalent merit	1

Sample answer:



$R\sin\theta = mg$
$R\cos\theta = mrw^2$

Dividing

$$\tan\theta = \frac{g}{rw^2}$$

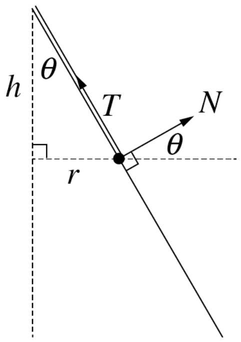
$$\frac{r}{h} = \frac{g}{rw^2}$$

$$w^2 = \frac{gh}{r^2}$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly resolves *N*, or equivalent merit	1

Sample answer:



Horizontal:

$$T\sin\theta - N\cos\theta = mrw^2$$
$$T\sin\theta\cos\theta - N\cos^2\theta = mrw^2\cos\theta \qquad —— ①$$

Vertical:

$$T\cos\theta + N\sin\theta = mg$$
$$T\sin\theta\cos\theta + N\sin^2\theta = mg\sin\theta \qquad —— ②$$

② – ① $N(\sin^2\theta + \cos^2\theta) = mg\sin\theta - mrw^2\cos\theta$

Question 14 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Observes $N \geq 0$, or equivalent merit	1

Sample answer:

As particle 2 moves in a circle of radius r it must have

$$N \geq 0$$

$$mg\left(\sin\theta - \frac{h}{r}\cos\theta\right) \geq 0$$

$$\sin\theta - \cot\theta\cos\theta \geq 0$$

$$\sin\theta \geq \cot\theta\cos\theta$$

$$\tan\theta \geq \cot\theta$$

In first quad so

$$\theta \geq \frac{\pi}{4}$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$I_1 = \int_0^1 x^1\sqrt{1-x^2}\;dx$$

$$= \frac{1}{2}\int_0^1 -2x\sqrt{1-x^2}\;dx$$

$$= -\frac{1}{2}\left[\frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= -\frac{1}{3}(0-1)$$

$$= \frac{1}{3}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly uses integration by parts, simplifying where possible, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

$$I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$$

Let $u = x^{n-1}$ $\quad dv = x\sqrt{1-x^2}dx$

$du = (n-1)x^{n-2}dx \quad v = -\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}$

$$\therefore I_n = \left[x^{n-1} \times \frac{-1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1 + \frac{1}{3}\int_0^1 \left(1-x^2\right)^{\frac{3}{2}}(n-1)x^{n-2}\, dx$$

$$= \frac{(n-1)}{3}\int_0^1 x^{n-2}\left(1-x^2\right)\sqrt{1-x^2}\, dx$$

$$= \frac{(n-1)}{3}\int_0^1 x^{n-2}\sqrt{1-x^2}\, dx - \frac{n-1}{3}\int_0^1 x^n\sqrt{1-x^2}\, dx$$

$$\therefore I_n = \frac{n-1}{3}I_{n-2} - \frac{n-1}{3}I_n$$

$$I_n\left(1+\frac{n-1}{3}\right) = \frac{n-1}{3}I_{n-2}$$

$$I_n\left(\frac{3+n-1}{3}\right) = \frac{n-1}{3}I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n+2}I_{n-2}$$

Question 15 (a) (iii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$
\begin{aligned}
I_5 &= \frac{4}{7} \times I_3 \\
&= \frac{4}{7} \times \frac{2}{5} \times I_1 \\
&= \frac{4}{7} \times \frac{2}{5} \times \frac{1}{3} \qquad \text{from part (i)} \\
&= \frac{8}{105}
\end{aligned}
$$

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Differentiates with respect to x, or equivalent merit	1

Sample answer:

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ (a constant)

Differentiating implicitly:

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{\frac{1}{2}} \times \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}$$

$$= -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$= -\sqrt{\frac{y}{x}}$$

$\therefore$ At P, gradient of tangent $= -\sqrt{\dfrac{d}{c}}$

and equation of tangent is

$$y - d = -\sqrt{\frac{d}{c}}(x - c)$$

$$\sqrt{c}\,y - \sqrt{c}\,d = -\sqrt{d}\,x + \sqrt{d}\,c$$

$$\therefore \quad y\sqrt{c} + x\sqrt{d} = d\sqrt{c} + c\sqrt{d}$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains an expression for $OA + OB$ in terms of c and d, or equivalent merit	2
• Finds an expression for OA, or equivalent merit	1

Sample answer:

At A, $y = 0$

$$\therefore \quad x\sqrt{d} = d\sqrt{c} + c\sqrt{d}$$

$$x = \frac{d\sqrt{c} + c\sqrt{d}}{\sqrt{d}}$$

$$\therefore \quad OA = \frac{d\sqrt{c} + c\sqrt{d}}{\sqrt{d}}$$

At B, $x = 0$

$$\therefore \quad y\sqrt{c} = d\sqrt{c} + c\sqrt{d}$$

$$\therefore \quad OB = \frac{d\sqrt{c} + c\sqrt{d}}{\sqrt{c}}$$

$$\begin{aligned}\therefore \quad OA + OB &= \frac{d\sqrt{c} + c\sqrt{d}}{\sqrt{d}} + \frac{d\sqrt{c} + c\sqrt{d}}{\sqrt{c}} \\ &= \sqrt{d}\sqrt{c} + c + d + \sqrt{c}\sqrt{d} \\ &= \sqrt{c}\left(\sqrt{d} + \sqrt{c}\right) + \sqrt{d}\left(\sqrt{d} + \sqrt{c}\right) \\ &= \left(\sqrt{c} + \sqrt{d}\right)\left(\sqrt{d} + \sqrt{c}\right)\end{aligned}$$

But P lies on $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$\therefore \sqrt{c} + \sqrt{d} = \sqrt{a}$$

$$\begin{aligned}\therefore OA + OB &= \sqrt{a} \times \sqrt{a} \\ &= a\end{aligned}$$

Question 15 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to solve the simultaneous equations, or equivalent merit	1

Sample answer:

At P, $\dfrac{x_1^{\,2}}{a^2} + \dfrac{y_1^{\,2}}{b^2} = 1$ and $\dfrac{x_1^{\,2}}{c^2} - \dfrac{y_1^{\,2}}{d^2} = 1$

$$\therefore \frac{x_1^{\,2}}{a^2} + \frac{y_1^{\,2}}{b^2} = \frac{x_1^{\,2}}{c^2} - \frac{y_1^{\,2}}{d^2}$$

$$x_1^{\,2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) = -y_1^{\,2}\left(\frac{1}{d^2} + \frac{1}{b^2}\right)$$

$$x_1^{\,2}\left(\frac{c^2 - a^2}{a^2c^2}\right) = -y_1^{\,2}\left(\frac{b^2 + d^2}{b^2d^2}\right)$$

$$\therefore \frac{x_1^{\,2}}{y_1^{\,2}} = -\left(\frac{b^2 + d^2}{b^2d^2}\right)\left(\frac{a^2c^2}{c^2 - a^2}\right)$$

$$= \frac{a^2c^2}{\left(a^2 - c^2\right)} \times \frac{\left(b^2 + d^2\right)}{b^2d^2}$$

Question 15 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Finds product of the two slopes, or equivalent merit	2
• Finds slope of tangent to ellipse, or equivalent merit	1

Sample answer:

Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{a^2} \times \frac{b^2}{y}$$

$\therefore$ Gradient of tangent at $P = \dfrac{-x_1 b^2}{a^2 y_1}$

Hyperbola: $\dfrac{x^2}{c^2} - \dfrac{y^2}{d^2} = 1$

Similarly gradient of tangent at $P = \dfrac{x_1}{c^2} \times \dfrac{d^2}{y_1}$

$$\therefore \text{Product of gradients} = \frac{-x_1 b^2}{a^2 y_1} \times \frac{x_1 d^2}{c^2 y_1}$$

$$= \frac{-{x_1}^2}{{y_1}^2} \times \frac{b^2 d^2}{a^2 c^2}$$

$$= \frac{-a^2 c^2}{\left(a^2 - c^2\right)} \times \frac{\left(b^2 + d^2\right)}{b^2 d^2} \times \frac{b^2 d^2}{a^2 c^2} \qquad \text{from part (i)}$$

$$= \frac{-\left(b^2 + d^2\right)}{a^2 - c^2}$$

But, conics have some foci so $ae = cE$

and $b^2 = a^2\left(1 - e^2\right)$ and $d^2 = c^2\left(E^2 - 1\right)$

$b^2 = a^2 - a^2 e^2$ $\qquad d^2 = c^2 E^2 - c^2$

$$\therefore b^2 + d^2 = a^2 - a^2 e^2 + c^2 E^2 - c^2$$

$$= a^2 - a^2 e^2 + a^2 e^2 - c^2$$

$$= a^2 - c^2$$

$$\begin{aligned}\therefore \text{ Product of gradients} &= \frac{-\left(b^2+d^2\right)}{a^2-c^2}\\ &= \frac{-\left(a^2-c^2\right)}{a^2-c^2}\\ &= -1\end{aligned}$$

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$\alpha = \cos\theta + i\sin\theta$

By de Moivre's theorem,

$$\alpha^k = \cos(k\theta) + i\sin(k\theta)$$

$$\alpha^{-k} = \cos(-k\theta) + i\sin(-k\theta)$$

$$= \cos(k\theta) - i\sin(k\theta)$$

$$\therefore \alpha^k + \alpha^{-k} = 2\cos k\theta$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Sums the series, multiplies top and bottom by $(1-\bar{\alpha})$ and attempts to expand, or equivalent merit	2
• Recognises that there are $2n + 1$ terms, or equivalent merit	1

Sample answer:

There are $(2n + 1)$ terms in the geometric series with first term α^{-n} and ratio α

$$C = \frac{\alpha^{-n}\left(1-\alpha^{2n+1}\right)}{(1-\alpha)}$$

$$= \frac{\alpha^{-n}\left(1-\alpha^{2n+1}\right)}{1-\alpha} \times \left(\frac{1-\bar{\alpha}}{1-\bar{\alpha}}\right)$$

$$= \frac{\alpha^{-n} - \alpha^{-(n+1)} - \alpha^{(n+1)} + \alpha^{n}}{(1-\alpha)(1-\bar{\alpha})}, \quad \text{using} \quad \bar{\alpha} = \frac{1}{\alpha}$$

$$= \frac{\left(\alpha^{n} + \alpha^{-n}\right) - \left(\alpha^{(n+1)} + \alpha^{-(n+1)}\right)}{(1-\alpha)(1-\bar{\alpha})}$$

Question 16 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Correctly matches appropriate terms and attempts to use the result from part (i) , or equivalent merit	1

Sample answer:

Write C as

$$\begin{aligned} C &= 1+\left(\alpha+\alpha^{-1}\right)+\left(\alpha^2+\alpha^{-2}\right)+\cdots+\left(\alpha^n+\alpha^{-n}\right) \\ &= 1+2\cos\theta+2\cos 2\theta+\cdots+2\cos(n\theta) \qquad \text{by part (i)} \\ &= 1+2\left(\cos\theta+\cos 2\theta+\cdots+\cos(n\theta)\right) \end{aligned}$$

$$\begin{aligned} \text{Also,}\quad C &= \frac{\alpha^n+\alpha^{-n}-\left(\alpha^{n+1}+\alpha^{-(n+1)}\right)}{(1-\alpha)(1-\bar{\alpha})} \\ &= \frac{2\cos n\theta-2\cos(n+1)\theta}{2-(\alpha+\bar{\alpha})} \\ &= \frac{2\cos n\theta-2\cos(n+1)\theta}{2-2\cos\theta} \\ &= \frac{\cos n\theta-\cos(n+1)\theta}{1-\cos\theta} \end{aligned}$$

Question 16 (a) (iv)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Put $\theta = \dfrac{\pi}{n}$, then

$$1+2\left(\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cdots+\cos\frac{n\pi}{n}\right) = \frac{\cos\left(\frac{n\pi}{n}\right)-\cos(n+1)\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}$$

$$RHS = \frac{-1-\cos\left(\pi+\frac{\pi}{n}\right)}{1-\cos\frac{\pi}{n}}$$

$$= \frac{-1+\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}$$

$$= -1$$

$$\therefore LHS = 1+2\left(\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cdots+\cos\frac{n\pi}{n}\right)$$

$$= -1$$

$$\therefore \cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cdots+\cos\left(\frac{n\pi}{n}\right) = -1$$

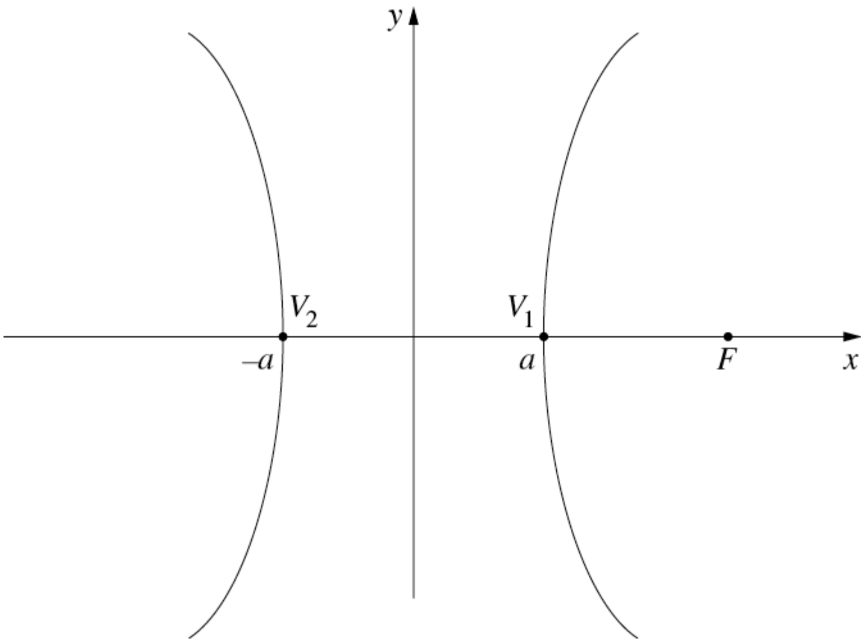
Which is independent of n.

Question 16 (b)

Criteria	Marks
• Provides correct solution	2
• Finds one correct value of a, or equivalent merit	1

Sample answer:

Since the eccentricity is 2 the conic is a hyperbola



The focus F is $(ae, 0) = (2a, 0)$

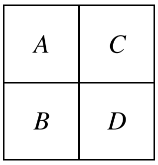
$V_1F = 1$, so $2a - a = 1 \Rightarrow a = 1$

$FV_2 = 2a + a = 1 \Rightarrow a = \frac{1}{3}$

Question 16 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains correct count for one case, or equivalent merit	1

Sample answer:



Case 1: Tiles B and C have the same colour.
There are $(x - 1)$ ways of choosing tile D.

Case 2: Tiles B and C have different colours.
There are $(x - 2)$ choices of colour for tile C
and there are $(x - 2)$ choices of colour for tile D
giving $(x - 2)^2$ choices in total.

$\therefore$ There are $(x - 2)^2 + (x - 1) = x^2 - 3x + 3$ choices in total.

Question 16 (c) (ii)

Criteria	Marks
• Provides correct proof	2
• Demonstrates true for $n = 1$	1

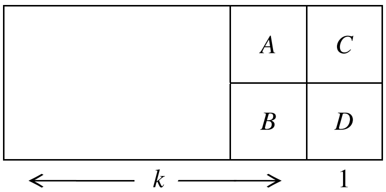
Sample answer:

Let $P(n)$ be the given proposition.

$P(1)$ is true since this was given.

Let k be an integer for which $P(k)$ is true.

That is, there are $x(x-1)(x^2 - 3x + 3)^{k-1}$ choices for the colours. Consider a $(k+1) \times 2$ grid.



As in part (i) there are $(x^2 - 3x + 3)$ choices for the colours of tiles C and D.

$\therefore$ There are $x(x-1)(x^2 - 3x + 3)^k$ choices for the colours.

$\therefore$ $P(k+1)$ is true, so $P(n)$ is true for all $n \geq 1$ by induction.

Question 16 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Finds the number of ways if not all colours need be used, or equivalent merit	1

Sample answer:

Let $G(x) = x(x-1)(x^2 - 3x + 3)^4$

The number of ways to colour the grid with 3 colours, each used at least once, is

$G(3) - 3 \times G(2)$ (there are 3 choices for which colour is left out)

$= 6 \times 3^4 - 3 \times 2 \times 1^4$

$= 480$

2017 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	2.4	E3
2	1	3.4	E3
3	1	2.2	E3
4	1	1	E6
5	1	7.5	E4
6	1	7.5	E3
7	1	1.9	E6
8	1	1	E9
9	1	6.3.1	E5
10	1	8	E6

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	2.2	E3
11 (a) (ii)	2	2.2	E3
11 (b)	2	3.2	E3
11 (c)	2	2.5	E3
11 (d)	3	4.1	E8
11 (e)	2	5.1	E7
11 (f)	3	4.1	E8
12 (a) (i)	1	8	E6
12 (a) (ii)	1	8	E6
12 (a) (iii)	1	8	E6
12 (a) (iv)	1	8	E6
12 (a) (v)	1	1.5	E6
12 (b)	3	2.1	E3
12 (c)	3	4.1	E8
12 (d) (i)	2	7.2	E4
12 (d) (ii)	2	7.2	E4
13 (a)	1	8.3	E2
13 (b) (i)	2	7.5	E4
13 (b) (ii)	2	8.3	E4

Question	Marks	Content	Syllabus outcomes
13 (c)	4	6.2.2	E5
13 (d)	4	5.1	E7
13 (e)	2	2.3	E3
14 (a) (i)	1	7.6	E8
14 (a) (ii)	2	4.1	E8
14 (a) (iii)	1	4.1	E9
14 (b) (i)	1	8.1(2.9)	E2
14 (b) (ii)	1	8.1(2.9)	E2
14 (b) (iii)	3	8.1(2.9)	E2
14 (c) (i)	2	6.3.4	E5
14 (c) (ii)	2	6.3.3	E5
14 (c) (iii)	2	6.3.3, 6.3.4	E5
15 (a) (i)	1	4.1	E8
15 (a) (ii)	3	4.1	E8
15 (a) (iii)	1	4.1	E8
15 (b) (i)	2	1.8	E6
15 (b) (ii)	3	8	E6
15 (c) (i)	2	3.1, 3.2	E4
15 (c) (ii)	3	3.1, 3.2	E4
16 (a) (i)	1	2.4	E3
16 (a) (ii)	3	2.1, 8	E3
16 (a) (iii)	2	2.1, 8	E3
16 (a) (iv)	1	8	E3
16 (b)	2	3.2	E3
16 (c) (i)	2	8	E2
16 (c) (ii)	2	8.2	E2
16 (c) (iii)	2	8	E2