BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HSC Mathematics Extension 2
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	C
3	A
4	D
5	B
6	C
7	D
8	C
9	A
10	B

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Provides exact argument in radians, or equivalent merit	1

Sample answer:

$z = \sqrt{3} - i$

$|z| = 2, \quad \operatorname{Arg}(z) = -\frac{\pi}{6}$

So $z = 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$z^6 = 2^6\left(\cos(-\pi) + i\sin(-\pi)\right)$

$= -2^6$ which is real

Question 11 (a) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

Take $n = 3$

$z^3 = 2^3\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$

$= -2^3 i$ which is purely imaginary

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

$$\int x\, e^{-2x}\, dx \qquad u = x, \quad \frac{dv}{dx} = e^{-2x}$$

$$= \frac{x e^{-2x}}{-2} + \frac{1}{2}\int e^{-2x}\, dx$$

$$= -\frac{x e^{-2x}}{2} - \frac{1}{4}e^{-2x} + C$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	2
• Correctly differentiates xy, or equivalent merit	1

Sample answer:

$x^3 + y^3 = 2xy$

Differentiating with respect to x,

$$3x^2 + 3y^2 y' = 2xy' + 2y$$

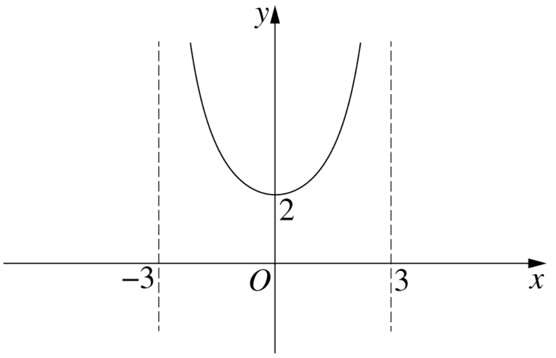
$$y'\left(3y^2 - 2x\right) = 2y - 3x^2$$

$$y' = \frac{2y - 3x^2}{3y^2 - 2x}$$

Question 11 (d) (i)

Criteria	Marks
• Provides correct graph	2
• Recognises correct domain, or equivalent merit	1

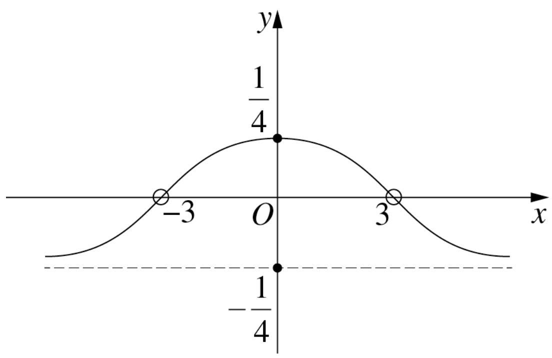
Sample answer:



Question 11 (d) (ii)

Criteria	Marks
• Provides correct graph	2
• Correctly deals with the asymptotes, or equivalent merit	1

Sample answer:



Question 11 (e)

Criteria	Marks
• Provides correct solution	2
• Provides correct domain, or equivalent merit	1

Sample answer:

$$f(x) = x\sin^{-1}\left(\frac{x}{2}\right)$$

Domain: $-2 \leq x \leq 2$

Range: $0 \leq f(x) \leq \pi$

Question 12 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Question 12 (a) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$$

Question 12 (a) (iii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

Foci at $\left(\pm ae,\ 0\right) = \left(\pm\sqrt{5},\ 0\right)$

Question 12 (a) (iv)

Criteria	Marks
• Provides correct answer	1

Sample answer:

Directrices are $x = \pm\dfrac{a}{e} = \pm\dfrac{9}{\sqrt{5}}$

Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$y = xf(x) - \int x\,f'(x)\,dx$$

$$\frac{dy}{dx} = f(x) + x\,f'(x) - xf'(x)$$

$$= f(x)$$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Attempts to apply result from part (i), or equivalent merit	1

Sample answer:

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int x\frac{1}{1+x^2}\,dx \qquad \text{(From part (i))}$$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log\left(1+x^2\right) + k$$

Question 12 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Applies de Moivre's theorem, or equivalent merit	1

Sample answer:

$$z = \cos\theta + i\sin\theta$$

$$z^4 = \cos 4\theta + i\sin 4\theta \qquad \text{(de Moivre's theorem)}$$

$$z^4 = (\cos\theta + i\sin\theta)^4$$

$\therefore$ equating real parts,

$$\cos 4\theta = \cos^4\theta - 6\cos^2\sin^2\theta + \sin^4\theta$$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

using $\sin^2\theta = 1 - \cos^2\theta$

$$\therefore \quad \cos 4\theta = \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

Alternative solution

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$\therefore \quad \cos 4\theta = 2\cos^2 2\theta - 1$$

$$= 2\left(2\cos^2\theta - 1\right)^2 - 1$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

Question 12 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Finds the slope of the normal at P, or equivalent merit	1

Sample answer:

$$xy = c^2$$

d.w.r.t. x $\quad y + x\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$= \frac{-c}{cp^2} = \frac{-1}{p^2}$$

Slope of normal at P is p^2.

So equation of normal is $y = p^2x + k$.

Since the normal passes through P

$$\frac{c}{p} = p^2 \times cp + k$$

$$k = \frac{c}{p} - cp^3$$

$$\therefore y = p^2x + \frac{c}{p} - cp^3$$

$$\frac{y}{p} = px + \frac{c}{p^2} - cp^2 \qquad (\div\ p)$$

$$\therefore px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right)$$

Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	3
• Obtains a quadratic whose roots are cp and cq, or equivalent merit	2
• Attempts to solve the equations for the normal and hyperbola, or equivalent merit	1

Sample answer:

Solving $xy = c^2$ and the equation of the normal,

$$px - \frac{c^2}{px} = c\left(p^2 - \frac{1}{p^2}\right)$$

$$\therefore \; p^2x^2 - pcx\left(p^2 - \frac{1}{p^2}\right) - c^2 = 0$$

The roots are cp and cq so using their product,

$$c^2pq = -\frac{c^2}{p^2}$$

$$\therefore \quad q = -\frac{1}{p^3}$$

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Finds the value of x at which there is a stationary point, or equivalent merit	2
• Obtains one of $\dfrac{f'(x)}{f(x)}$ or $1 + \ln x$, or equivalent merit	1

Sample answer:

$\log f(x) = x \log x$

Differentiating w.r.t. x $\quad \dfrac{f'(x)}{f(x)} = 1 + \log x$

Now $f(x) > 0$, so stationary points occur when $1 + \log x = 0$.

$\therefore\ x = \dfrac{1}{e}$. Check gradients each side.

x	$\dfrac{1}{e}^-$	$\dfrac{1}{e}$	$\dfrac{1}{e}^+$
$f'(x)$	–	0	+

So there is a local minimum at $x = \dfrac{1}{e}$

Alternatively, $\quad f'(x) = (1 + \log x) f(x)$

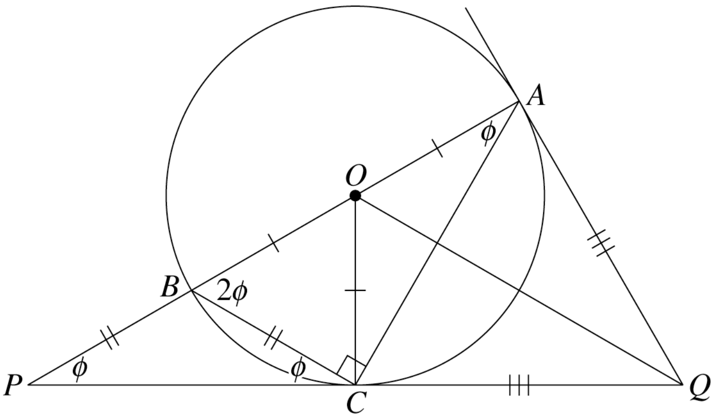
so $\quad f''(x) = \dfrac{1}{x} f(x) + (1 + \log x) f'(x)$

and $f''\left(\dfrac{1}{e}\right) = e f\left(\dfrac{1}{e}\right) > 0$ so $x = \dfrac{1}{e}$ gives a minimum.

Question 13 (b)

Criteria	Marks
• Provides correct solution	4
• Makes substantial progress	3
• Shows $\angle BAC = \angle BPC$, or equivalent merit	2
• Shows $\angle OBC$ is twice $\angle OPC$, or equivalent merit	1

Sample answer:



Construct OC, AC, OQ

Let $\angle BPC$ have size ϕ
$\triangle BPC$ is isosceles so $\angle BCP$ is ϕ
$\angle OBC$ is an external angle
So $\angle OBC = 2\phi$

$\angle BAC$ is angle on chord BC
So $\angle BAC = \phi$ (Angle in alt. seg. and angle between chord and tangent)

But $\triangle BAC$ has a right angle at C (Angle in semicircle)

So $\phi + 2\phi = \dfrac{\pi}{2}$

$\phi = \dfrac{\pi}{6}$

$OA \perp AQ$ (Angle between tangent and radius)

So $\angle CAQ = \dfrac{\pi}{3}$

$QA = QC$ (Tangents from point equal)

So $\angle AQC = \dfrac{\pi}{3}$ ($\triangle QAC$ isosceles)

Hence $\angle AQC = \dfrac{\pi}{3}$ (Angle sum of $\triangle$)

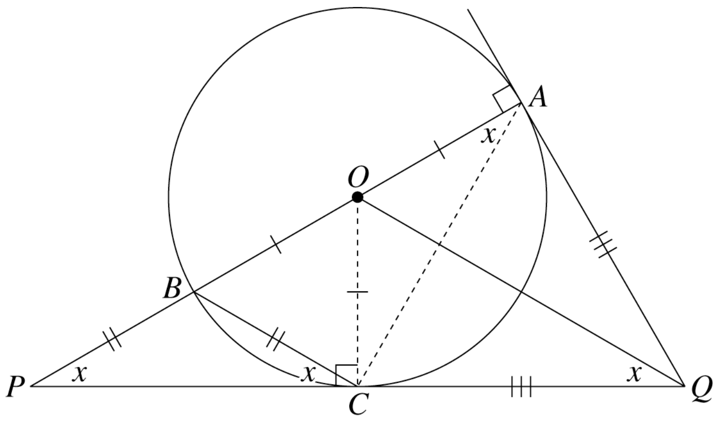
$\triangle OAQ$ and $\triangle OCQ$ are cong. (sss, $OA = OC$ radii, $AQ = CQ$ above, OQ common)

So $\angle OQC = \dfrac{1}{2} \times \dfrac{\pi}{3} = \dfrac{\pi}{6} = \angle OPC$

Thus, $\triangle OPQ$ is isosceles
And, $OP = OQ$

Question 13 (b) (continued)

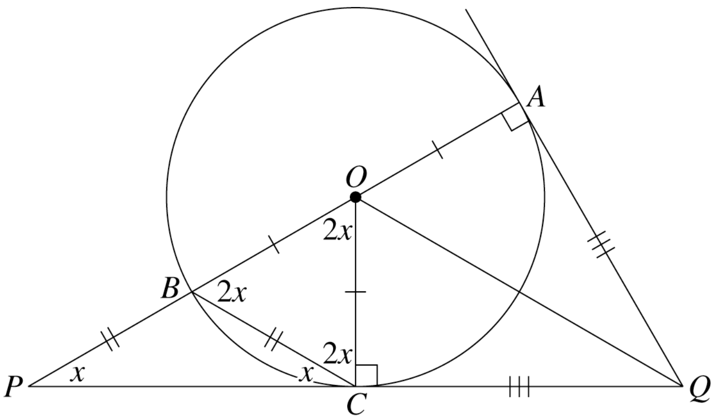
Alternative solution



$OP = OQ$	Let $\angle BPC = x$	
	$\therefore\ \angle BCP = x$	(Base $\angle$s of isosceles $\triangle$)
	Construct AC	
	$\angle BAC = x$	(Alt. seg. thm.)
	Hence, $AQ \perp OA$ and $OC \perp PQ$	(Tangents to circle)
	$OAQC$ is cyclic quad.	(Opp. $\angle$s supplementary)
	$\therefore\ \angle OAC = x$	(Angles standing on same arc)
	Hence, $OP = OQ$	(Base $\angle$s of isosceles $\triangle$ are equal)

Question 13 (b) (continued)

Alternative solution



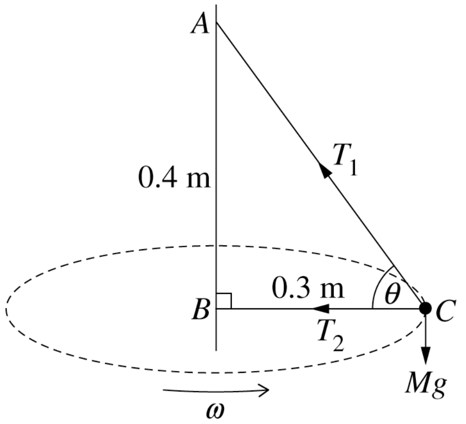
Let $\angle BPC = x$

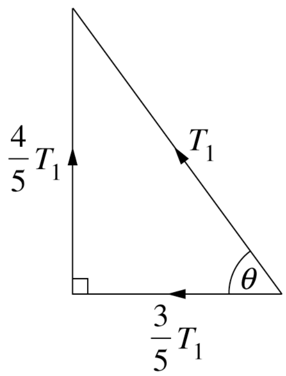
$\angle BPC = \angle PCB = x$	(Base $\angle$ isosceles $\triangle$)
$\angle OBC = 2x$	(Exterior $\angle$ $\triangle PBC$)
$\therefore\ \angle OCB = 2x$	(Equal $\angle$s of isosceles $\triangle OBC$, $OB = OC$ radii)
$\therefore\ \angle OCP = \dfrac{\pi}{2} = 3x$	($\angle$ between radius and tangent)
$\therefore\ x = \dfrac{\pi}{6}$	
$\therefore\ \triangle OBC$ is equilateral	($\angle$s = 60°)
$\therefore\ \triangle OCQ \equiv \triangle OAQ$	($OC = OA$ radii, OQ common, $AQ = CQ$)
$\angle AOC = 4x$	(Exterior angle)
$\angle COQ = \angle AOQ = 2x$	($\triangle OCQ \equiv \triangle OAQ$)
$\therefore\ \triangle OPC \equiv \triangle OQC$	(Equiangular, OC common)
$\therefore\ OP = OQ$	

Question 13 (c) (i)

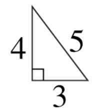
Criteria	Marks
• Provides correct solution	3
• Attempts to eliminate T_1, or equivalent merit	2
• Resolves T_1 into components, or equivalent merit	1

Sample answer:





$\theta = \tan^{-1}\frac{4}{3}$



$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

Resolving vertically: $\frac{4}{5}T_1 = Mg \qquad (g = 10)$

$$T_1 = \frac{5}{4}M(10)$$

$$\boxed{T_1 = \frac{25M}{2}}$$

Resolving horizontally: $T_2 + \frac{3}{5}T_1 = M(0.3)\omega^2$

$$\therefore\ T_2 + \frac{3}{5} + \frac{25M}{2} = \frac{3}{10}M\omega^2$$

$$T_2 = \frac{3}{10}M\omega^2 - \frac{3}{10}(25M)$$

$$\boxed{T_2 = \frac{3}{10}M\left(\omega^2 - 25\right)}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$T_2 > T_1 \quad \Rightarrow \quad \frac{3}{10}\cancel{M}\left(\omega^2 - 25\right) > \frac{25}{2}\cancel{M}$$

$$3\omega^2 - 75 > 125$$

$$3\omega^2 > 200$$

$$\omega^2 > \frac{200}{3}$$

$$|\omega| > \sqrt{\frac{200}{3}} \qquad \omega > 0$$

$$\omega > 10\sqrt{\frac{2}{3}}$$

Question 13 (d) (i)

Criteria	Marks
• Provides correct solution	2
• Differentiates $p(x)$, or equivalent merit	1

Sample answer:

$$p(x) = ax^3 + bx^2 + cx + d$$

$$p'(x) = 3ax^2 + 2bx + c$$

$$\Delta = (2x)^2 - 4(3a)(c)$$

$$= 4b^2 - 12ac$$

$$= 4\left(b^2 - 3ac\right)$$

$$< 0 \text{ if } b^2 - 3ac < 0 \text{ as given.}$$

$\therefore$ $p'(x)$ has no real solutions.

$\therefore$ $p(x)$ has no stationary points.

Hence $p(x)$ is either an increasing or decreasing function for all x.

$\therefore$ $p(x)$ cuts the x-axis only once.

Question 13 (d) (ii)

Criteria	Marks
• Provides correct solution	2
• Shows $p'\left(-\frac{b}{3a}\right)=0$, or equivalent merit	1

Sample answer:

$$p\left(-\frac{b}{3a}\right)=0$$

$$\begin{aligned}p'\left(-\frac{b}{3a}\right) &= 3a\left(-\frac{b}{3a}\right)^2+2b\left(-\frac{b}{3a}\right)+c\\ &= \frac{b^2}{3a}-\frac{2b^2}{3a}+c\\ &= -\frac{1}{3a}\left(b^2-3ac\right)\\ &= 0\end{aligned}$$

So at least a double root

$$p''(x)=6ax+2b$$

$$\begin{aligned}p''\left(-\frac{b}{3a}\right) &= 6a\left(-\frac{b}{3a}\right)+2b\\ &= -2b+2b\\ &= 0\end{aligned}$$

So $x=-\frac{b}{3a}$ is a triple root.

<u>Alternatively</u>, $b^2=3ac \quad \Rightarrow \quad c=\frac{b^2}{3a}$

$$p(x)=ax^3+bx^2+\frac{b^2x}{3a}+d$$

$$p\left(-\frac{b}{3a}\right)=0$$

$$\Rightarrow \quad -\frac{ab^3}{27a^3}+\frac{b^3}{9a^2}-\frac{b^3}{9a^2}+d=0$$

$$\Rightarrow \quad d=\frac{b^3}{27a^2}$$

$$\begin{aligned}\therefore \quad 27a^2\,p(x) &= 27a^3x^3+27a^2b^2x^2+9ab^2x+b^3\\ &= (3ax+b)^3\end{aligned}$$

$\therefore$ $p(x)$ has a triple root at $x=-\frac{b}{3a}$.

Question 14 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\int \sin^3 x \, dx$$

$$= \int \sin x \left(1 - \cos^2 x\right) dx$$

$$= \int \sin x - \sin x \cos^2 x \, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

<u>OR</u> $\dfrac{d}{dx}\left(\dfrac{1}{3}\cos^3 x - \cos x + C\right)$

$$= \frac{1}{3} \cdot 3\cos^2 x(-\sin x) - (-\sin x) + 0$$

$$= -\cos^2 x \sin x + \sin x$$

$$= \sin x\left(-\cos^2 x + 1\right)$$

$$= \sin\left(\sin^2 x\right)$$

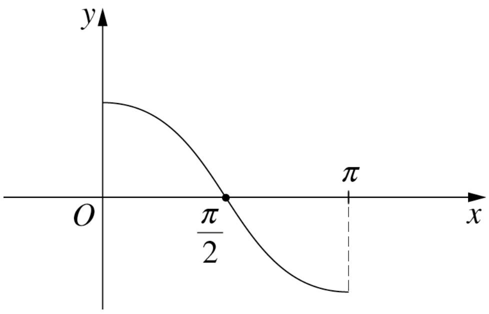
$$= \sin^3 x$$

$$\therefore \int \sin^3 x \, dx = \frac{1}{3}\cos^3 x - \cos x + C$$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct explanation	1

Sample answer:



$$\int_0^{\pi} \cos x \, dx = 0 \text{ by symmetry}$$

Since $2n - 1$ is odd,

$$\int_0^{\frac{\pi}{2}} \cos^{2n-1} x \, dx = -\int_{\frac{\pi}{2}}^{\pi} \cos^{2n-1} x \, dx$$

$$\therefore \int_0^{\pi} \cos^{2n-1} x \, dx = 0$$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct solution	3
• Attempts to evaluate correct integral using integration by parts, or equivalent merit	2
• Obtains correct expression for the volume, or equivalent merit	1

Sample answer:

By method of shells,

$$\text{Volume} = 2\pi \int x f(x)\,dx$$

$$= 2\pi \int_0^{\pi} x \sin^3 x\,dx$$

Using integration by parts and part (i)

$$\text{Volume} = 2\pi \left(\left[x\left(\frac{1}{3}\cos^3 x - \cos x \right) \right]_0^{\pi} - \int_0^{\pi} \left(\frac{1}{3}\cos^3 x - \cos x \right) dx \right)$$

The integral above is 0 by part (ii), so

$$\text{Volume} = 2\pi \left(\pi\left(-\frac{1}{3} + 1 - 0 \right) - 0 \right)$$

$$= \frac{4}{3}\pi^2$$

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	3
• Obtains an integral that can be easily evaluated, or equivalent merit	2
• Chooses suitable substitution, or equivalent merit	1

Sample answer:

$$I_0 = \int_0^1 \frac{1}{\left(x^2+1\right)^2}\,dx$$

Let $x = \tan\theta$, then $\dfrac{dx}{d\theta} = \sec^2\theta$, so

$$I_0 = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\Bigg|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$I_0 + I_2 = \int_0^1 \frac{1+x^2}{\left(x^2+1\right)^2}\,dx$$

$$= \int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \tan^{-1}x\Big|_0^1 = \frac{\pi}{4}$$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Obtains the value of $I_2 + I_4$, or equivalent merit	2
• Attempts to evaluate $I_2 + I_4$, or equivalent merit	1

Sample answer:

Consider $I_2 + I_4$

$$= \int_0^1 \frac{x^2 + x^4}{\left(x^2+1\right)^2}\,dx$$

$$= \int_0^1 \frac{x^2}{x^2+1}\,dx$$

$$= \int_0^1 1 - \frac{1}{x^2+1}\,dx$$

$$= x - \tan^{-1}x\Big|_0^1$$

$$= 1 - \frac{\pi}{4}$$

$$\therefore\ I_4 = 1 - \frac{\pi}{4} - I_2$$

$$= 1 - \frac{\pi}{4} - \left(\frac{\pi}{4} - I_0\right)$$

$$= 1 - \frac{\pi}{2} + \left(\frac{\pi}{8} + \frac{1}{4}\right)$$

$$= \frac{5}{4} - \frac{3\pi}{8}$$

Question 14 (c)

Criteria	Marks
• Provides correct solution	3
• Observes $(x-1)(\sqrt{x}-1) \geq 0$ for $x \geq 1$, or equivalent merit	2
• Investigates $x\sqrt{x}+1-x-\sqrt{x}$, or equivalent merit	1

Sample answer:

For $x \geq 0$, Required to show that $x\sqrt{x}+1 \geq x+\sqrt{x}$

Now
$$\begin{aligned} x\sqrt{x}+1-\left(x+\sqrt{x}\right) &= x\sqrt{x}-\sqrt{x}-x+1 \\ &= \sqrt{x}(x-1)-(x-1) \\ &= \left(x-1\right)\left(\sqrt{x}-1\right) \end{aligned}$$

For $0 \leq x < 1$, $(x-1) < 0$, $\left(\sqrt{x}-1\right) < 0$ $(-)(-)$

$\therefore\ x\sqrt{x}+1-\left(x+\sqrt{x}\right) > 0$

For $x \geq 1$, $(x-1) \geq 0$, $\left(\sqrt{x}-1\right) \geq 0$ $(+)(+)$

$\therefore\ x\sqrt{x}+1-\left(x+\sqrt{x}\right) \geq 0$

Hence for $x \geq 0$, $x\sqrt{x}+1 \geq x+\sqrt{x}$

<u>Alternative solution</u>

For $x \geq 0$, Required to show that $x\sqrt{x}+1 \geq x+\sqrt{x}$

Now
$$\begin{aligned} x\sqrt{x}+1-\left(x+\sqrt{x}\right) &= x\sqrt{x}-\sqrt{x}-x+1 \\ &= \sqrt{x}(x-1)-(x-1) \\ &= \left(x-1\right)\left(\sqrt{x}-1\right) \\ &= \left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)\left(\sqrt{x}-1\right) \\ &= \left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)^2 \\ &\geq 0 \end{aligned}$$

Since $\sqrt{x}+1 \geq 0$ for $x \geq 0$

and $\left(\sqrt{x}-1\right)^2 \geq 0$ for real $x \geq 0$

$\therefore\ x\sqrt{x}+1 \geq x+\sqrt{x}$

Question 15 (a)

Criteria	Marks
• Provides correct solution	2
• Obtains the value of $\alpha^2+\beta^2+\gamma^2$, or equivalent merit	1

Sample answer:

$x^3 - 3x + 1 = 0$ has roots α, β, γ, then

$$\alpha+\beta+\gamma = 0$$
$$\alpha\beta+\alpha\gamma+\beta\gamma = -3$$
$$\alpha\beta\gamma = -1$$

Let $A = \alpha^2$, $B = \beta^2$, $C = \gamma^2$

$$A+B+C = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) = 6$$
$$AB+AC+BC = \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$$
$$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$$
$$= 9$$
$$ABC = \alpha^2\beta^2\gamma^2 = 1$$

$\therefore$ A, B, C satisfy $x^3 - 6x^2 + 9x - 1$

<u>Alternative solution</u>

Let $y = x^2$ then, $x = \sqrt{y}$

So $\left(\sqrt{y}\right)^3 - 3\sqrt{y} + 1 = 0$

$\Rightarrow$ $y\sqrt{y} - 3\sqrt{y} = -1$

$$\sqrt{y}(y-3) = -1$$
$$y(y-3)^2 = 1$$

$\Rightarrow$ $y^3 - 6y^2 + 9y - 1 = 0$

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly applies $a = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$, or equivalent merit	1

Sample answer:

$$\ddot{x} = -\frac{\mu^2}{x^2}$$

$$\frac{1}{2}\frac{d}{dx}\left(v^2\right) = -\frac{\mu^2}{x^2}$$

$$\Rightarrow \qquad v^2 = \frac{2\mu^2}{x} + C$$

$$x = b, \quad v = 0 \quad \Rightarrow \quad C = -\frac{2\mu^2}{b}$$

$$\therefore \quad v^2 = 2\mu^2\left(\frac{1}{x} - \frac{1}{b}\right)$$

$$= 2\mu^2\left(\frac{b-x}{bx}\right)$$

$\therefore \quad v = -\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$, negative root since the particle moves to the left.

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly applies given substitution to integrand, or equivalent merit	2
• Obtains integral for *t* in terms of *x*, or equivalent merit	1

Sample answer:

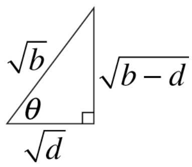
$$\frac{dx}{dt} = -\mu\sqrt{2}\sqrt{\frac{b-x}{bx}}$$

$$t = -\frac{1}{\mu\sqrt{2}}\int_b^d \sqrt{\frac{bx}{b-x}}\,dx$$

Put $x = b\cos^2\theta$, $\dfrac{dx}{d\theta} = -2b\cos\theta\sin\theta$

$x = b \quad \Rightarrow \quad \theta = 0$

$x = d \quad \Rightarrow \quad \theta = \cos^{-1}\sqrt{\dfrac{d}{b}}$



Then $t = \dfrac{2b\sqrt{b}}{\sqrt{2}\mu}\displaystyle\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\sin\theta\,d\theta$

$$= \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta\,d\theta$$

Question 15 (b) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\lim_{d \to 0^+} t = \frac{1}{\mu}\sqrt{\frac{b}{2}} \cdot \frac{b\pi}{2}$$

$$= \frac{\pi}{2\sqrt{2}\mu} b\sqrt{b}$$

Question 15 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Uses method of partial fractions to compare numerators, or equivalent merit	1

Sample answer:

Write $\dfrac{3!}{x(x+1)(x+2)(x+3)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x+2} + \dfrac{D}{x+3}$

$\therefore\ 3! = A(x+1)(x+2)(x+3) + Bx(x+2)(x+3) + Cx(x+1)(x+3) + Dx(x+1)(x+2)$

$x = 0 \quad \Rightarrow \quad A = 1$

$x = -1 \quad \Rightarrow \quad -2B = 6 \quad \Rightarrow \quad B = -3$

$x = -2 \quad \Rightarrow \quad 2C = 6 \quad \Rightarrow \quad C = 3$

$x = -3 \quad \Rightarrow \quad -6D = 6 \quad \Rightarrow \quad D = -1$

Question 15 (c) (ii)

Criteria	Marks
• Provides correct solution	3
• Uses $x = -k$ to obtain an expression for a_k, or equivalent merit	2
• Uses method of partial fractions to compare numerators, or equivalent merit	1

Sample answer:

$$\frac{n!}{x(x+1)\ldots(x+n)} = \frac{a_0}{x} + \frac{a_1}{x+1} + \cdots + \frac{a_k}{x+k} + \cdots + \frac{a_n}{x+n}$$

Put RHS over common denominator, then general term is

$$a_k x(x+1)\ldots(x+k-1)(x+k+1)\ldots(x+n)$$

At $x = -k$, all terms are zero except,

$$-k(-k+1)\ldots(-1)(1)2\ldots(-k+n)a_k \equiv n!$$

$$(-1)^k[1\cdot 2\ldots k][1\cdot 2\ldots(n-k)]a_k = n!$$

$$\therefore a_k = (-1)^k \frac{n!}{k!(n-k)!}$$

$$= (-1)^k \binom{n}{k}$$

Question 15 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Identifies $x = 1$ using part (ii), or equivalent merit	1

Sample answer:

Put $x = 1$ in part (ii) then

$$\frac{1}{1} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + \frac{(-1)^n}{n+1} = \frac{n!}{1\cdot 2\ldots(n+1)}$$

$$= \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

Hence the limiting sum $= \lim_{n\to\infty} \frac{1}{n+1} = 0$.

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	3
• Finds θ or α, or equivalent merit	2
• Uses real and imaginary parts or equivalent merit	1

Sample answer:

$z = \cos\theta + i\sin\theta$

$w = \cos\alpha + i\sin\alpha$

$1 + z + w = 0$

$\Rightarrow \quad (1 + \cos\theta + \cos\alpha) + i(\sin\theta + \sin\alpha) = 0$

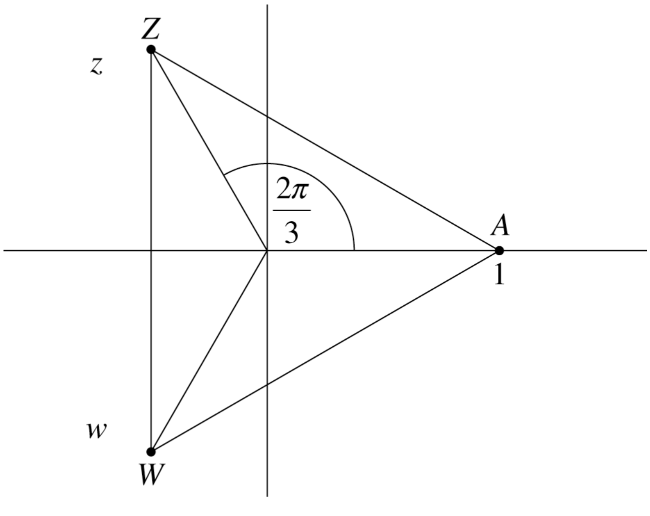
Equating real and imaginary parts,

$$1 + \cos\theta + \cos\alpha = 0 \qquad (1)$$
$$\sin\theta = -\sin\alpha \qquad (2)$$

(2) $\Rightarrow \quad \theta = -\alpha$

(1) $\Rightarrow \quad \cos\theta = -\dfrac{1}{2}$

$\therefore \quad \theta = \dfrac{2\pi}{3}, \quad \alpha = -\dfrac{2\pi}{3}$



Since $|z| = |w| = 1$

$\angle ZAW = \dfrac{\pi}{3}$

So the triangle is equilateral.

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Divides by one of $2i$, z_1, z_2, or equivalent merit	1

Sample answer:

Dividing each complex number by $2i$, we have $1, \dfrac{z_1}{2i}, \dfrac{z_2}{2i}$, which have unit modulus and sum 0.

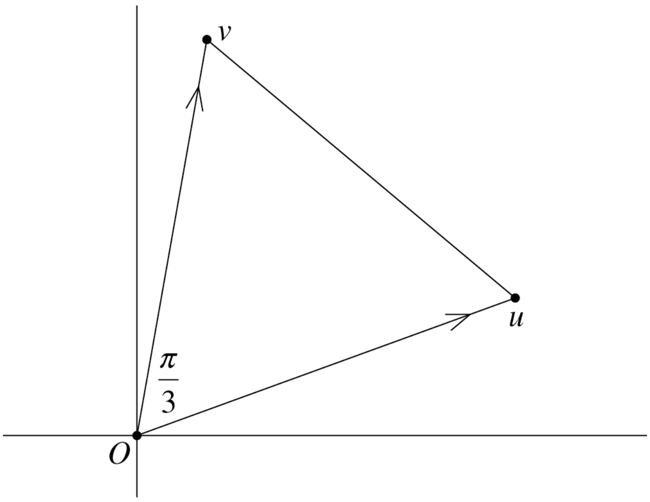
So by part (i) they form the vertices of an equilateral triangle.

Multiplying by $2i$ rotates the triangle by $\dfrac{\pi}{2}$ and dilates the sides by a factor of 2, giving a similar triangle. Hence $2i$, z_1, z_2 form the vertices of an equilateral triangle.

Question 16 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Observes that v is u rotated by $\pm\dfrac{\pi}{3}$	1

Sample answer:



The vector O_v is vector O_u rotated about O anticlockwise through $\dfrac{\pi}{3}$, so

$$v = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)u$$

$$\therefore \quad v^3 = -u^3$$

$$\Rightarrow \quad (v+u)\left(u^2 + v^2 - uv\right) = 0$$

$$v \neq -u, \text{ so } u^2 + v^2 = uv$$

Question 16 (b) (ii)

Criteria	Marks
• Provides a correct example	1

Sample answer:

For example, take $u = 1$ then

$1 + v^2 = v$, so

$v^2 - v + 1 = 0$

$$v = \frac{1+\sqrt{-3}}{2} = \frac{1+\sqrt{3}i}{2}$$

Alternative Take $u = 1$

then can use

$$v = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$= \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Question 16 (c) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

There are $(n-1)$ people who can swap hats with Tom. Removing Tom and this person then $(n-2)$ people are left, giving $D(n-2)$ derangements.

Hence there are $(n-1)\,D(n-2)$ such derangements.

Question 16 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Identifies remaining case, or equivalent merit	1

Sample answer:

There are $(n-1)$ people who can take Tom's hat. Suppose Tom has not directly swapped a hat with exactly one person. One person takes Tom's hat and leaves. Tom takes their place and we have a derangement of $(n-1)$ people. Hence in this case there are $(n-1)\,D(n-1)$ derangements.

Hence the total number of derangements is $D(n)=(n-1)\left[D(n-1)+D(n-2)\right]$.

Alternative

Ignoring Tom, there are $D(n-1)$ derangements of the remaining hats. If Tom swaps hats with any of the $(n-1)$ others we have a suitable derangement. A total of $(n-1)\,D(n-1)$ derangements.

Tom either swaps hats with one ———— $(n-1)\,D(n-2)$

or he doesn't ———— $(n-1)\,D(n-1)$

Total $D(n)=(n-1)\left[D(n-1)+D(n-2)\right]$

Question 16 (c) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

From part (ii),

$$D(n)=(n-1)\left[D(n-1)+D(n-2)\right]$$

$$\therefore \quad D(n)=nD(n-1)-D(n-1)+(n-1)D(n-2)$$

$$\therefore \quad D(n)-nD(n-1)=-\left[D(n-1)-(n-1)D(n-2)\right]$$

Question 16 (c) (iv)

Criteria	Marks
• Provides correct solution	1

Sample answer:

By part (iii),

$$\begin{aligned} D(n) - nD(n-1) &= -\left[D(n-1) - (n-1)D(n-2)\right] \\ &= (-1)^2\left[D(n-2) - (n-2)D(n-3)\right] \\ &= (-1)^3\left[D(n-3) - (n-3)D(n-4)\right] \\ &\quad \vdots \\ &= (-1)^{n-2}\left[D(2) - D(1)\right] \\ &= (-1)^{n-2}[1-0] \\ &= (-1)^{n-2} \times (-1)^2 \\ &= (-1)^n \end{aligned}$$

Question 16 (c) (v)

Criteria	Marks
• Provides correct solution	2
• Makes some progress	1

Sample answer:

When $n = 1$

$$1!\sum_{r=0}^{1}\frac{(-1)^r}{r!} = 1!(1-1) = 0 = D(1)$$

So statement is true for $n = 1$.

Let k be an integer such that $D(k) = k!\sum_{r=0}^{k}\frac{(-1)^r}{r!}$

Then

$$D(k+1) = (k+1)D(k) + (-1)^{k+1} \qquad \text{(By part (iv))}$$

$$= (k+1)!\sum_{r=0}^{k}\frac{(-1)^r}{r!} + (-1)^{k+1}\cdot\frac{(k+1)!}{(k+1)!}$$

$$= (k+1)!\sum_{r=0}^{k+1}\frac{(-1)^r}{r!}$$

$\therefore$ Statement is true for $n = k+1$, so true for all $n \geq 1$ by induction.

<u>Alternative solution</u>

From part (iv),

$$\frac{1}{n!}D(n) - \frac{1}{(n-1)!}D(n-1) = \frac{(-1)^n}{n!}$$

Substituting $n = 2, 3, \ldots$ and adding we have

$$\frac{1}{n!}D(n) - D(1) = \sum_{r=2}^{n}\frac{(-1)^r}{r!}$$

Now since $\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} = 0$ we have

$$D(n) = n!\sum_{r=0}^{n}\frac{(-1)^r}{r!}$$

2016 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	1.3	E6
2	1	7.2	E4
3	1	3.4	E3
4	1	2.1, 2.2, 2.3	E3
5	1	2.1, 2.2	E3
6	1	7.4, 8 (16.3E)	PE3, E4
7	1	3.3	E4
8	1	6.3	E5
9	1	5.1	E7
10	1	2.4	E3

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	2	2.2	E3
11 (a) (ii)	1	2.4	E3
11 (a) (iii)	1	2.4	E3
11 (b)	3	4.1	E8
11 (c)	2	1.8	E6
11 (d) (i)	2	1.7	E6
11 (d) (ii)	2	1.5	E6
11 (e)	2	8 (15E)	HE4, E6
12 (a) (i)	1	3.1	E3
12 (a) (ii)	1	3.1	E3
12 (a) (iii)	1	3.1	E3
12 (a) (iv)	1	3.1	E3
12 (b) (i)	1	8 (8.8)	PE5, E2
12 (b) (ii)	2	4.1	E8
12 (c) (i)	2	2.1, 2.4, 8 (17.1E)	E3
12 (c) (ii)	1	8	E3
12 (d) (i)	2	3.3	E4
12 (d) (ii)	3	3.3	E4
13 (a)	3	1.8, 8 (12.4)	E6
13 (b)	4	8 (2.10E)	E2, PE3

Question	Marks	Content	Syllabus outcomes
13 (c) (i)	3	6.3.3	E5
13 (c) (ii)	1	6.3.3	E5
13 (d) (i)	2	7.5	E4
13 (d) (ii)	2	7.2, 7.5	E4
14 (a) (i)	1	4.1	E8
14 (a) (ii)	1	1.6, 1.9	E6
14 (a) (iii)	3	5.1	E7
14 (b) (i)	3	4.1	E8
14 (b) (ii)	1	4.1	E8
14 (b) (iii)	3	4.1	E8
14 (c)	3	8.3	E4
15 (a)	2	7.5	E4
15 (b) (i)	2	6.1	E5
15 (b) (ii)	3	4.1	E8
15 (b) (iii)	1	6.1	E2
15 (c) (i)	2	7.6	E8
15 (c) (ii)	3	7.6	E8
15 (c) (iii)	2	8 (17.3E)	E4
16 (a) (i)	3	2.1, 2.2	E3
16 (a) (ii)	2	2.1, 2.2	E3
16 (b) (i)	2	2.1, 2.2	E3
16 (b) (ii)	1	2.1, 2.2	E3
16 (c) (i)	1	8 (18E)	E2
16 (c) (ii)	2	8 (18E)	E2
16 (c) (iii)	1	8 (18E)	E2
16 (c) (iv)	1	8 (18E)	E2
16 (c) (v)	2	8 (18E)	E2