2022 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	C
2	A
3	D
4	A
5	B
6	B
7	C
8	D
9	D
10	B

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j}, \qquad \underset{\sim}{v} = 2\underset{\sim}{i} + \underset{\sim}{j}$

$$\begin{aligned} \underset{\sim}{u} + 3\underset{\sim}{v} &= \underset{\sim}{i} - \underset{\sim}{j} + 3\left(2\underset{\sim}{i} + \underset{\sim}{j}\right) \\ &= \underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{i} + 3\underset{\sim}{j} \\ &= 7\underset{\sim}{i} + 2\underset{\sim}{j} \end{aligned}$$

Question 11 (a) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{v} &= 1 \times 2 + (-1) \times 1 \\ &= 1 \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains correct integrand, or equivalent merit	2
• Attempts to use the substitution, or equivalent merit	1

Sample answer:

$$\int_0^1 \frac{x}{\sqrt{x^2+4}}\,dx$$

$$= \int_0^1 \frac{2x}{2\sqrt{x^2+4}}\,dx$$

$$= \int_4^5 \frac{1}{2\sqrt{u}}\,du$$

$$= \left[\sqrt{u}\,\right]_4^5$$

$$= \sqrt{5} - 2$$

$$u = x^2 + 4$$

$$du = 2x\,dx$$

when $x = 0$, $u = 4$

when $x = 1$, $u = 5$

Question 11 (c)

Criteria	Marks
• Provides correct solution	2
• Obtains a correct expression for one coefficient, or equivalent merit	1

Sample answer:

$$\left(1-\frac{x}{2}\right)^8 = \cdots + \binom{8}{2}(1)^6\left(-\frac{x}{2}\right)^2 + \binom{8}{3}(1)^5\left(-\frac{x}{2}\right)^3 + \cdots$$

$$= \cdots + 28 \times \frac{x^2}{4} + 56 \times \frac{-x^3}{8} + \cdots$$

$$= \cdots + 7x^2 - 7x^3 + \cdots$$

$\therefore$ Coefficient of x^2 is 7.

And coefficient of x^3 is –7.

Question 11 (d)

Criteria	Marks
• Provides correct solution	2
• Observes that $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$, or equivalent merit	1

Sample answer:

$$\underset{\sim}{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}, \qquad \underset{\sim}{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$$

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular $\therefore\ \underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

That is $\quad a(a-7) + 2(4a-1) = 0$

$$a^2 - 7a + 8a - 2 = 0$$
$$a^2 + a - 2 = 0$$
$$(a+2)(a-1) = 0$$
$$\therefore \quad a = -2 \quad \text{or} \quad 1$$

Question 11 (e)

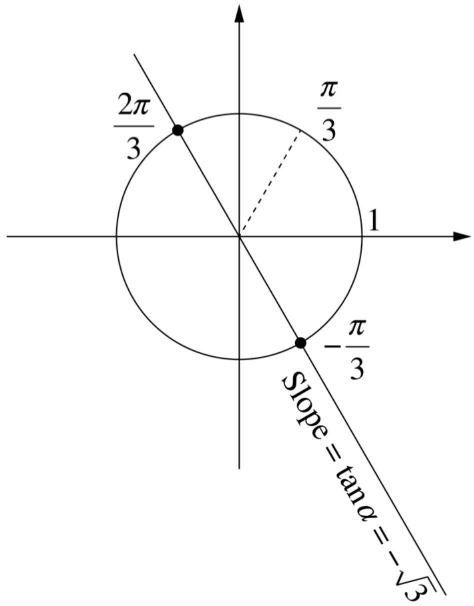
Criteria	Marks
• Provides correct solution	3
• Finds correct argument, or equivalent merit	2
• Finds the correct value of R, or equivalent merit	1

Sample answer:

$$\sqrt{3}\sin(x) - 3\cos(x) = R\sin(x+\alpha)$$
$$= R\sin x\cos\alpha + R\cos x\sin\alpha$$

$\therefore \quad R\sin\alpha = -3 \qquad (1)$

$\quad R\cos\alpha = \sqrt{3} \qquad (2)$



(1) ÷ (2): $\quad \tan\alpha = \dfrac{-3}{\sqrt{3}}$

$$= -\sqrt{3}$$

$\therefore \quad \alpha = -\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ (See unit circle)

Since $\cos\alpha > 0$ and $\sin\alpha < 0$

$$\alpha = -\frac{\pi}{3}$$

From (1) $\quad R^2\sin^2\alpha = 9 \qquad (3)$

From (2) $\quad R^2\cos^2\alpha = 3 \qquad (4)$

$(3)+(4)$: $\quad R^2\left(\sin^2\alpha + \cos^2\alpha\right) = 12$

$$R^2 = 12$$
$$R = 2\sqrt{3}$$

$$\therefore \sqrt{3}\sin(x) - 3\cos(x) = 2\sqrt{3}\sin\left(x - \frac{\pi}{3}\right)$$

Question 11 (f)

Criteria	Marks
• Provides correct solution	3
• Identifies the correct critical values, or equivalent merit	2
• Excludes $x = 2$ OR attempts to deal with the denominator, or equivalent merit	1

Sample answer:

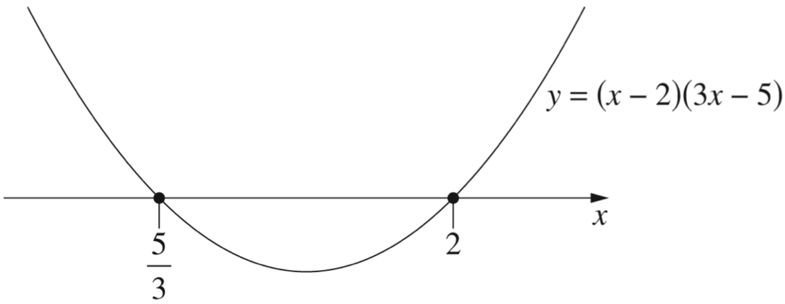
$$\frac{x}{2-x} \geq 5 \qquad x \neq 2$$

$$x(2-x) \geq 5(2-x)^2$$

$$(2-x)[5(2-x)-x] \leq 0$$

$$(2-x)(-6x+10) \leq 0$$

$$(x-2)(3x-5) \leq 0$$

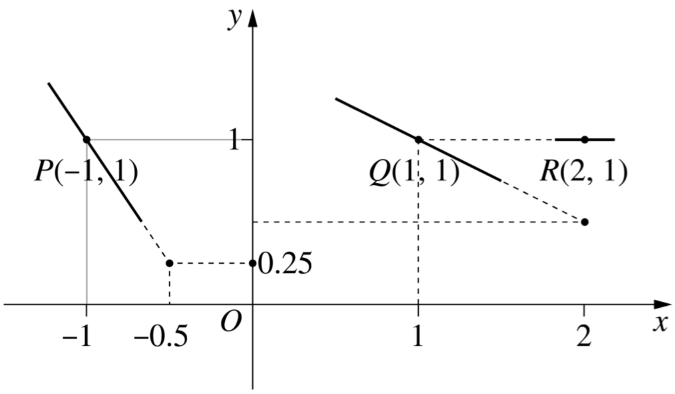


Solutions $= \left[\frac{5}{3},\ 2\right)$

Question 12 (a)

Criteria	Marks
• Provides correct drawing	2
• Shows one correct slope	1

Sample answer:



At $P(-1, 1)$ $\quad \dfrac{dy}{dx} = \dfrac{-1-2}{1+1} = -\dfrac{3}{2} = -\dfrac{0.75}{0.5}$

At $Q(1, 1)$ $\quad \dfrac{dy}{dx} = \dfrac{1-2}{1+1} = -\dfrac{1}{2} = -\dfrac{0.5}{1}$

At $R(2, 1)$ $\quad \dfrac{dy}{dx} = \dfrac{2-2}{4+1} = 0$

Question 12 (b)

Criteria	Marks
• Provides correct solution	2
• Attempts to use the pigeonhole principle, or equivalent merit	1

Sample answer:

$$\frac{\text{Number of players above limit}}{\text{Number of teams}} = \frac{41}{13} > 3$$

$\therefore$ Using the Pigeonhole principle, at least one team has more than 3 players above limit. At least one team will be penalised.

Question 12 (c)

Criteria	Marks
• Provides correct solution	3
• Finds the slope of the tangent, or equivalent merit	2
• Attempts to find the derivative of $x \arctan x$, or equivalent merit	1

Sample answer:

$$y = x\tan^{-1}(x)$$

$$y' = x \times \frac{1}{1+x^2} + \tan^{-1}(x) \times 1$$

$$= \frac{x}{1+x^2} + \tan^{-1}(x)$$

At $\left(1, \frac{\pi}{4}\right)$ $\quad y' = \frac{1}{2} + \tan^{-1}(1)$

$$= \frac{1}{2} + \frac{\pi}{4}$$

$$= \frac{\pi + 2}{4}$$

Equation of tangent

$$y - \frac{\pi}{4} = \left(\frac{\pi+2}{4}\right)\left(x - 1\right)$$

$$4y - \pi = (\pi + 2)(x - 1)$$

$$4y = (\pi + 2)x - \pi - 2 + \pi$$

$$4y = (\pi + 2)x - 2$$

$$\therefore \quad y = \frac{(\pi+2)}{4}x - \frac{1}{2}$$

Question 12 (d) (i)

Criteria	Marks
• Provides correct solution	3
• Integrates to obtain T as a function involving an exponential function, or equivalent merit	2
• Separates the variables in the differential equation, or equivalent merit	1

Sample answer:

$$\frac{dT}{dt} = k\left(T - T_1\right)$$

Note that as $t \to +\infty$, $\frac{dT}{dt} \to 0$ and T approaches room temperature so T_1 = room temperature = 12.

Also T decreases with time, so $T - T_1 > 0$.

$\frac{dT}{T-12} = k\,dt$ (separate variables)

$\ln(T - 12) = kt + c$ where c is a constant

$T - 12 = \mathrm{A}e^{kt}$ where $A = e^c$

$T = 12 + \mathrm{A}e^{kt}$

When $t = 0$: $92 = 12 + A$ so $A = 80$

When $t = 5$: $76 = 12 + 80e^{5k}$

$$e^{5k} = \frac{4}{5}$$

$$5k = \ln\left(\frac{4}{5}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{4}{5}\right)$$

Finally $T = 12 + 80e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$ for $t \geq 0$.

Question 12 (d) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

For $T = 57$

$$57 = 12 + 80e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$$

$$e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t} = \frac{57-12}{80} = \frac{9}{16}$$

$$\frac{1}{5}\ln\left(\frac{4}{5}\right)t = \ln\left(\frac{9}{16}\right)$$

$$t = \frac{5\ln\left(\frac{9}{16}\right)}{\ln\left(\frac{4}{5}\right)} = 12.89\ldots \qquad \approx 13 \text{ minutes}$$

The coffee will reach a temperature of 57°C after about 13 minutes.

Question 12 (e)

Criteria	Marks
• Provides correct solution	2
• Attempts to use result associated with the binomial distribution, or equivalent merit	1

Sample answer:

Let p be the proportion of red balls and n the number of trials.

Expected number of red balls $= np = 4 \times \frac{3}{10}$

Expected number of green balls $= n(1-p) = 4 \times \frac{7}{10}$

Expected score $= 10 \times 4 \times \frac{3}{10} - 5 \times 4 \times \frac{7}{10} = 12 - 14 = -2$.

Answers could include:

- The expected score if you pick <u>one</u> ball is $\frac{3 \times 10 + 7 \times (-5)}{10} = -0.5$.
- It is the same for every one of the 4 balls since the balls are replaced so the expected score for the game is $4 \times (-0.5) = -2$.

Question 12 (f)

Criteria	Marks
• Provides correct proof	3
• Proves the inductive step, or equivalent merit	2
• Establishes the base case, or equivalent merit	1

Sample answer:

METHOD 1

Let $a_n = 15^n + 6^{2n+1}$

Base case: $n = 0$

$a_0 = 1 + 6 = 7$ is divisible by 7

so the property holds for $n = 0$

Assume a_k is divisible by 7

$a_k = 15^k + 6^{2k+1} = 7M$, for some integer M.

Prove true for a_{k+1}.

$a_{k+1} = 15^{k+1} + 6^{2k+3} = 7Q$, for some integer Q.

$$
\begin{aligned}
LHS &= 15\left(15^k\right) + 6^{2k+3} \\
&= 15\left(7M - 6^{2k+1}\right) + 6^2 \times 6^{2k+1} \qquad \text{(from assumption)} \\
&= 105M - 15 \times 6^{2k+1} + 36 \times 6^{2k+1} \\
&= 105M + 21 \times 6^{2k+1} \\
&= 7\left(15M + 3 \times 6^{2k+1}\right) \\
&= 7Q \\
&= RHS
\end{aligned}
$$

This proves, using mathematical induction, that for all integers $n \geq 0$, $15^n + 6^{2n+1}$ is divisible by 7.

Answers could include:

METHOD 2
Let $a_n = 15^n + 6^{2n+1}$

Base case: $n = 0$

$a_0 = 1 + 6 = 7$ is divisible by 7

so the property holds for $n = 0$

Assume a_k is divisible by 7

$$\begin{aligned} a_{k+1} &= 15^{k+1} + 6^{2k+3} \\ &= 15\left(15^k\right) + 6^{2k+3} \end{aligned}$$

Substituting $15^k = a_k - 6^{2k+1}$

$$\begin{aligned} a_{k+1} &= 15\left(a_k - 6^{2k+1}\right) + 6^{2k+3} \\ &= 15a_k + 6^{2k+1}(-15 + 36) \\ &= 15a_k + 21 \times 6^{2k+1} \end{aligned}$$

a_k and 21 are both divisible by 7

so a_{k+1} is also divisible by 7

This proves, using mathematical induction, that for all integers $n \geq 0$, $15^n + 6^{2n+1}$ is divisible by 7.

Question 13 (a)

Criteria	Marks
• Provides correct solution	3
• Attempts to find the dot product of $\overrightarrow{BH}$ and $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or equivalent merit	2
• Write $\overrightarrow{BH}$ or $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, or equivalent merit	1

Sample answer:

We have $\overrightarrow{BH} = \underset{\sim}{h} - \underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{c}$, while $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$. Hence

$$
\begin{aligned}
\overrightarrow{BH} \cdot \overrightarrow{CA} &= \left(\underset{\sim}{a} + \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{a} - \underset{\sim}{c}\right) \\
&= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{c} \\
&= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{c} \\
&= |\underset{\sim}{a}|^2 - |\underset{\sim}{c}|^2 \\
&= 0 \qquad \text{(as the lengths of } \underset{\sim}{a} \text{ and } \underset{\sim}{c} \text{ are equal, as they are radii of a circle)}
\end{aligned}
$$

Hence, the two vectors are perpendicular.

Answers could include:

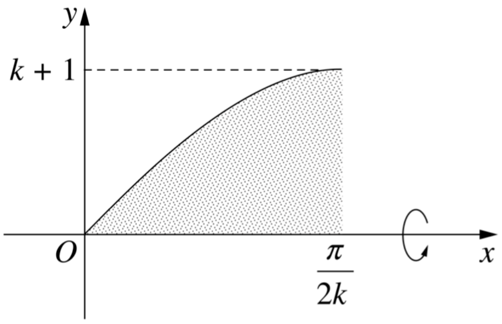
We may observe that $\underset{\sim}{a} + \underset{\sim}{c}$ and $\underset{\sim}{a} - \underset{\sim}{c}$ form the diagonals of a rhombus, as $\underset{\sim}{a}$ and $\underset{\sim}{c}$ have equal length. The diagonals of a rhombus are perpendicular and so $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

Question 13 (b)

Criteria	Marks
• Provides correct solution	3
• Obtains primitive in terms of $\sin(2kx)$, or equivalent merit	2
• Finds an integral expression for the volume of the solid in terms of $\sin(kx)$, or equivalent merit	1

Sample answer:

$y = (k+1)\sin(kx)$



$$V = \pi\int_0^{\frac{\pi}{2k}} (k+1)^2 \sin^2(kx)\cdot dx$$

$$= \pi(k+1)^2\int_0^{\frac{\pi}{2k}} \frac{1-\cos 2kx}{2}\cdot dx$$

$$= \frac{\pi(k+1)^2}{2}\int_0^{\frac{\pi}{2k}} 1-\cos 2kx\cdot dx$$

$$= \frac{\pi(k+1)^2}{2}\left[x-\frac{\sin 2kx}{2k}\right]_0^{\frac{\pi}{2k}}$$

$$= \frac{\pi(k+1)^2}{2}\left(\frac{\pi}{2k}-\frac{\sin\pi}{2k}\right)$$

$$= \frac{\pi^2(k+1)^2}{4k}$$

$$\cos 2\theta = 1-2\sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\therefore\ \sin^2(kx) = \frac{1}{2}(1-\cos 2kx)$$

For $V = \pi^2$,

$$\frac{(k+1)^2}{4k} = 1$$

$\therefore\quad k = 1\qquad$ (by inspection)

Or solve $k^2 + 2k + 1 = 4k$

$$(k-1)^2 = 0$$

$$k = 1$$

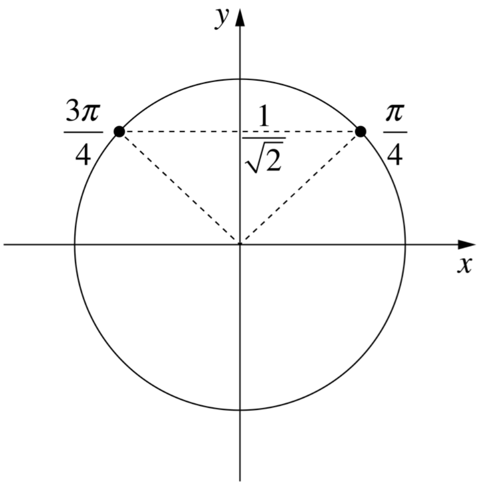
Question 13 (c)

Criteria	Marks
• Provides correct solution	2
• Finds a value of x where $g(f(x)) \neq x$	1

Sample answer:

The domain of $g(f(x))$ is $\mathbb{R}$.
$g(f(x)) = \arcsin(\sin x)$

If g is the inverse function of f, then $g(f(x))$ is equal to x for all x in the domain of $g(f(x))$ by definition of an inverse function.



Let $x = \frac{3\pi}{4}$
$\arcsin\left(\sin\frac{3\pi}{4}\right) = \frac{\pi}{4}$, see unit circle above
So for $x = \frac{3\pi}{4}$, $\arcsin(\sin x) \neq x$

So it is not true that $g(f(x)) = x$ for all x in the domain $g(f(x))$. So, g is NOT the inverse of f.

Question 13 (d)

Criteria	Marks
• Provides correct solution	3
• Evaluates $P'(\alpha)+P'(\beta)+P'(\gamma)$, or equivalent merit	2
• Writes $\alpha^2+\beta^2+\gamma^2$ in term of $\alpha+\beta+\gamma$ and so on OR Defines $P(x)$ and evaluates $P'(\alpha)$ OR Equivalent merit	1

Sample answer:

$$\alpha^2+\beta^2+\gamma^2=85$$
$$P'(\alpha)+P'(\beta)+P'(\gamma)=87$$

Let $P(x)=x^3+ax^2+bx+c$

Then $a=-(\alpha+\beta+\gamma)$

$b=\alpha\beta+\beta\gamma+\gamma\alpha$

$c=-\alpha\beta\gamma$

$$P'(x)=3x^2+2ax+b$$

$$\begin{aligned}
&P'(\alpha)+P'(\beta)+P'(\gamma)\\
&\quad=3\left(\alpha^2+\beta^2+\gamma^2\right)+2a(\alpha+\beta+\gamma)+3b\\
&\quad=3\left(\alpha^2+\beta^2+\gamma^2\right)-2(\alpha+\beta+\gamma)^2+3(\alpha\beta+\beta\gamma+\gamma\alpha)\\
&\quad=3\left(\alpha^2+\beta^2+\gamma^2\right)-2\left(\alpha^2+\beta^2+\gamma^2\right)-4(\alpha\beta+\beta\gamma+\gamma\alpha)+3(\alpha\beta+\beta\gamma+\gamma\alpha)\\
&\quad=\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\gamma\alpha)
\end{aligned}$$

$$\begin{aligned}
87&=85-(\alpha\beta+\beta\gamma+\gamma\alpha)\\
\alpha\beta+\beta\gamma+\gamma\alpha&=85-87\\
&=-2
\end{aligned}$$

Question 13 (e) (i)

Criteria	Marks
• Provides correct solution	2
• Introduces a suitable normal approximation, or equivalent merit	1

Sample answer:

METHOD 1

Let $\hat{p}$ be the proportion of chocolate bars which weigh less than 150 g.

$$\mathcal{P} = P(\hat{p} \geq 0.5)$$

Approximate $\hat{p}$ by a normal distribution with the same mean as $\hat{p}$ and the same standard deviation as $\hat{p}$.

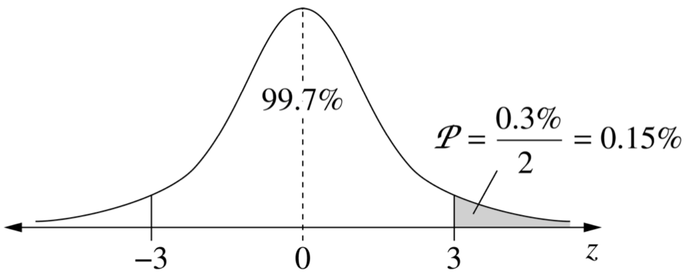
$$E(\hat{p}) = 0.2$$

$$\sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{16}} = 0.1$$

$$\begin{aligned} \mathcal{P} &= P\left(\frac{\hat{p}-0.2}{0.1} \geq \frac{0.5-0.2}{0.1}\right) \\ &\approx P(Z \geq 3) \end{aligned}$$

where Z follows a standard normal distribution.

Using known values



So $\mathcal{P} = P(Z \geq 3) = 0.15\% = 0.0015$

Alternatively using the table of values for the normal distribution, we get the more precise value of $\mathcal{P} = 0.13\% = 0.0013$

Answers could include:

METHOD 2

Let $\hat{p}$ be the proportion of bars in a sample of 16 bars which weigh <u>more</u> than 150 g.

$$\mathcal{P} = P(\hat{p} < 0.5)$$

(If 50% or more of the bars weigh less than 150 g, the remaining ones weigh 150 g or more.)

$$E(\hat{p}) = 0.8$$

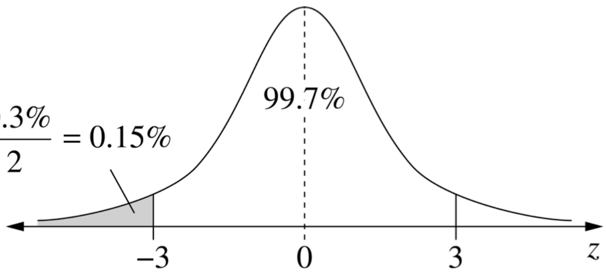
$$\sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{16}} = 0.1$$

$$\begin{aligned}\mathcal{P} &= P(\hat{p} < 0.5)\\ &= P\left(\frac{\hat{p}-0.8}{0.1} < \frac{0.5-0.8}{0.1}\right)\\ &\approx P(Z < -3)\end{aligned}$$

When we approximate $\hat{p}$ by a normal distribution.

$$\mathcal{P} \simeq P(Z < -3) \approx \frac{0.3\%}{2} = 0.15\% \text{ using known values}$$

$$\mathcal{P} = \frac{0.3\%}{2} = 0.15\%$$



(and $\mathcal{P} = 0.13\%$ using the table of normal values)

METHOD 3: (Using counts rather than proportion, without continuity correction)

Let X be the number of chocolate bars in a sample of 16 which weigh less than 150 g.

$\mathcal{P} = P(X \geq 8)$

X follows a binomial distribution $\text{Bin}(n, p) = \text{Bin}(16, 0.2)$, assuming the factory manager's claim is correct.

$$E(X) = np = 16 \times 0.2 = 3.2$$
$$\sigma(X) = \sqrt{np(1-p)} = \sqrt{16 \times 0.2 \times 0.8} = 1.6$$

$$\begin{aligned}\mathcal{P} &= P(X \geq 8)\\ &= P\left(\frac{X-3.2}{1.6} \geq \frac{8-3.2}{1.6}\right)\\ &\approx P(Z \geq 3)\end{aligned}$$

If we approximate X by a normal distribution (without the continuity correction) we get

$\mathcal{P} \approx P(Z \geq 3)$ where Z follows a standard normal distribution

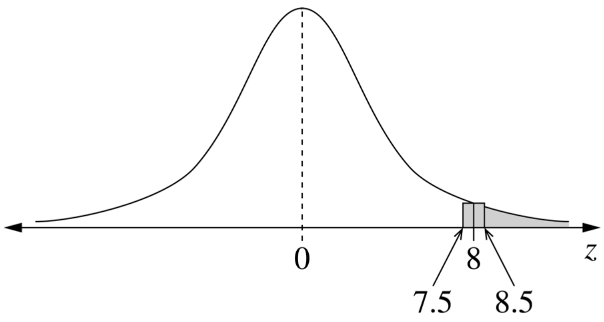
So $\mathcal{P} \approx 0.15\%$ using known values

<u>METHOD 4</u>: (Using <u>counts</u> and the continuity correction.)

Same X as method 3.

We approximate X by a normal distribution Y with the same mean and standard deviation as X.

$$\begin{aligned}\mathcal{P} &= P(X \geq 8) \\ &= P(Y > 7.5)\end{aligned}$$



$$\begin{aligned}\mathcal{P} &= P\left(\frac{Y-3.2}{1.6} > \frac{7.5-3.2}{1.6}\right) \\ &\approx P(Z > 2.6875) && \text{where } Z \text{ is a standard normal} \\ &= 1 - P(Z \leq 2.6875) && \text{using the nearest value from the table} \\ &\approx 1 - 0.9964 \\ &= 0.0036 \\ &= 0.36\%\end{aligned}$$

Question 13 (e) (ii)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

The number of items checked may not be large enough to assume the data is normally distributed.

Question 14 (a)

Criteria	Marks
• Provides correct solution	4
• Evaluates the constant of integration to find a correct equation for $\ln\lvert y \rvert$	3
• Obtains correct primitive, or equivalent merit	2
• Separates the variables in the differential equation, or equivalent merit	1

Sample answer:

$(x-2)\dfrac{dy}{dx} = xy$ and $(0, 1)$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x-2}\,dx$$

$$\ln\lvert y\rvert = \int 1 + \frac{2}{x-2}\,dx$$

$$= x + 2\ln\lvert x-2\rvert + C$$

$$= \ln\lvert x-2\rvert^2 + x + C$$

$$\ln\left(\frac{\lvert y\rvert}{(x-2)^2}\right) = x + C$$

$$\frac{\lvert y\rvert}{(x-2)^2} = e^{x+C}$$

$$\lvert y\rvert = e^{x+C}(x-2)^2$$

$$y = ke^x(x-2)^2 \qquad \text{where} \quad k = \pm e^C$$

When $x = 0$, $y = 1$ $\qquad \therefore \quad k = \dfrac{1}{4}$

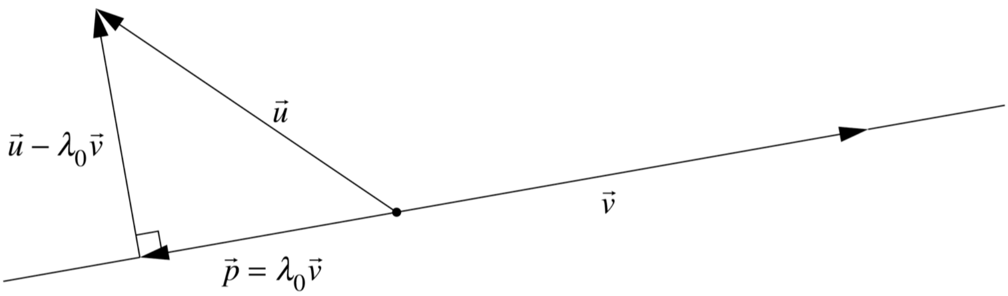
$$y = \frac{1}{4}e^x(x-2)^2$$

Question 14 (b)

Criteria	Marks
• Provides correct solution	3
• Uses an appropriate vector method to obtain a suitable inequality, or equivalent merit	2
• Writes $\underset{\sim}{u} - \lambda \underset{\sim}{v}$ as a sum of vectors parallel and perpendicular to $\underset{\sim}{v}$, or equivalent merit	1

Sample answer:

METHOD 1



Let λ be a real number.

$$\begin{aligned}
\left|\vec{u} - \lambda\vec{v}\right|^2 &= \left|\vec{u} - \lambda_0\vec{v} + \lambda_0\vec{v} - \lambda\vec{v}\right|^2 \\
&= \left|\underbrace{\left(\vec{u} - \lambda_0\vec{v}\right)}_{\text{Perpendicular to } \vec{v}} + \underbrace{\left(\lambda_0 - \lambda\right)\vec{v}}_{\text{Parallel to } \vec{v}}\right|^2 \\
&= \left[\left(\vec{u} - \lambda_0\vec{v}\right) + \left(\lambda_0 - \lambda\right)\vec{v}\right] \cdot \left[\left(\vec{u} - \lambda_0\vec{v}\right) + \left(\lambda_0 - \lambda\right)\vec{v}\right] \\
&= \left|\vec{u} - \lambda_0\vec{v}\right|^2 + \left|\left(\lambda_0 - \lambda\right)\vec{v}\right|^2 + 0 + 0
\end{aligned}$$

$$\left|\left(\lambda_0 - \lambda\right)\vec{v}\right|^2 \geq 0$$

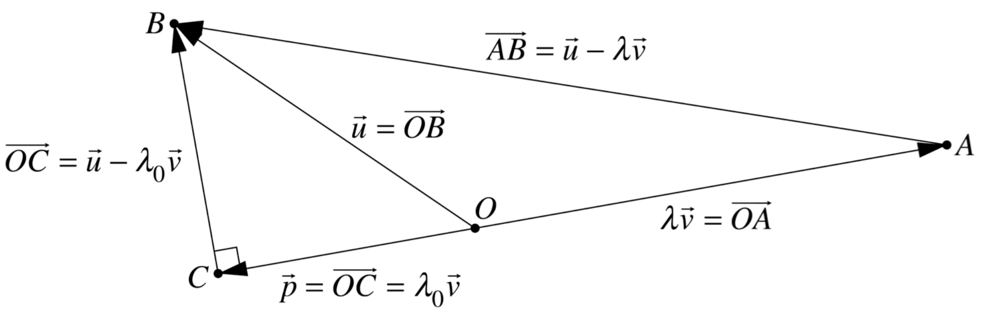
Therefore $\left|\vec{u} - \lambda\vec{v}\right|^2 \geq \left|\vec{u} - \lambda_0\vec{v}\right|^2$,

So $\left|\vec{u} - \lambda\vec{v}\right| \geq \left|\vec{u} - \lambda_0\vec{v}\right|$.

Hence $\left|\vec{u} - \lambda\vec{v}\right|$ is smallest when it equals $\left|\vec{u} - \lambda_0\vec{v}\right|$, and so is smallest when $\lambda = \lambda_0$.

Answers could include:

METHOD 2



Let $\lambda \in \mathbb{R}$.

Let O be a point in the plane.

Let A, B and C be the points defined by

$\overrightarrow{OA} = \lambda\vec{v}$

$\overrightarrow{OB} = \vec{u}$

$\overrightarrow{OC} = \vec{p}$

Because $\vec{p}$ is the projection of $\vec{u}$ onto $\vec{v}$, the triangle ABC has a right angle at C.

In any right-angled triangle the length of the hypotenuse is greater than or equal to the length of any of the sides

so $\left|\overrightarrow{AB}\right| \geq \left|\overrightarrow{BC}\right|$

That is $\left|\vec{u} - \lambda\vec{v}\right| \geq \left|\vec{u} - \lambda_0\vec{v}\right|$.

Hence $\left|\vec{u} - \lambda\vec{v}\right|$ is smallest when it equals $\left|\vec{u} - \lambda_0\vec{v}\right|$, and so is smallest when $\lambda = \lambda_0$.

Question 14 (c)

Criteria	Marks
• Provides correct solution	4
• Finds an expression for d in terms of the maximum range, or equivalent merit	3
• Finds maximum range, or equivalent merit	2
• Attempts to find the time of flight of projectile, or equivalent merit	1

Sample answer:

For the player to hit the target there must be a time t_1 where the positions of the target and of the projectile coincide:

$$\begin{cases} 2ut_1\cos\theta = d + ut_1 & \text{(i)} \\ 2ut_1\sin\theta = \dfrac{g}{2}t_1^{\,2} & \text{(ii)} \end{cases}$$

By (i) $\quad t_1 = \dfrac{d}{2u\cos\theta - u}$

t_1 is not 0, so dividing (ii) by t_1 yields

$$2u\sin\theta = \frac{g}{2}t_1 \qquad \text{(iii)}$$

Substitute in t_1 in (iii):

$$2u\sin\theta = \frac{g}{2}\left(\frac{d}{2u\cos\theta - u}\right)$$

$$d = \frac{2u\sin\theta\big(u\times(2\cos\theta - 1)\big)\times 2}{g}$$

$$d = \frac{4u^2}{g}\sin\theta(2\cos\theta - 1)$$

<u>Maximum range</u>:

It occurs at time t_2 and corresponds to $2ut_2\sin\theta - \dfrac{g}{2}t_2^{\,2} = 0$.

$t_2 \neq 0 \quad$ so $\quad 2u\sin\theta - \dfrac{g}{2}t_2 = 0$

$$t_2 = \frac{4u\sin\theta}{g}$$

The range is $\quad 2ut_2\cos\theta = \dfrac{8u^2}{g}\sin\theta\cos\theta$

$$= \frac{4u^2}{g}\sin 2\theta$$

This is maximum when $\sin 2\theta$ is maximum, that is when $\sin 2\theta = 1$.

The maximum range is $\dfrac{4u^2}{g}$.

Find the maximum value of $\sin\theta(2\cos\theta - 1)$ for θ in $\left(0, \dfrac{\pi}{2}\right)$.

$\sin\theta(2\cos\theta - 1) = \sin 2\theta - \sin\theta$

Let f be the function defined on $\left(0, \dfrac{\pi}{2}\right)$

by $\quad f(\theta) = \sin 2\theta - \sin\theta$

$$f'(\theta) = 2\cos 2\theta - \cos\theta$$

$$= 2\left(2\cos^2\theta - 1\right) - \cos\theta$$

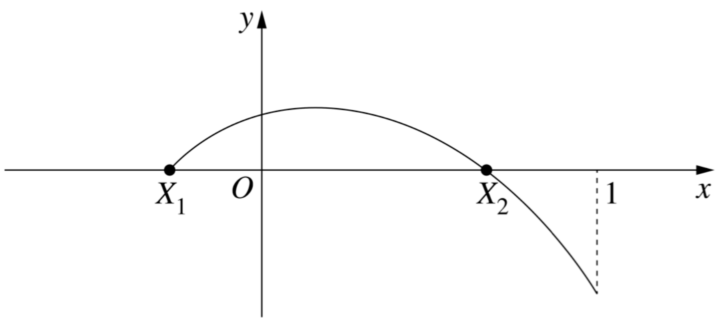
Let $X = \cos\theta$

$f'(\theta) = 2\left(2X^2 - 1\right) - X = 4X^2 - X - 2$

$\Delta = b^2 - 4ac = 1 - 4(-8) = 33$

$X_1 = \dfrac{1 - \sqrt{33}}{8}$ and $X_2 = \dfrac{1 + \sqrt{33}}{8}$

$\approx -0.59 \qquad\qquad \approx 0.84$



$f'(\theta) \geq 0$ if $0 \leq X \leq X_2$

$f'(\theta) \leq 0$ if $X_2 \leq X \leq 1$

So $f(\theta)$ is maximum when $X = X_2$

That is $\cos\theta = \dfrac{1 + \sqrt{33}}{8}$

With $0 < \theta < \dfrac{\pi}{2}$, that means $\theta \approx 0.5678$ which corresponds to $\sin 2\theta - \sin\theta = 0.3690\ldots \simeq 37\%$.

$$d = \underbrace{\frac{4u^2}{g}}_{\text{maximum range}} \times \underbrace{\sin\theta(2\cos\theta - 1)}_{\leq 37\%}$$

So d must be less than 37% of the maximum range.

Question 14 (d)

Criteria	Marks
• Provides correct solution	4
• Obtains a quadratic inequality in N, where N is the number of tickets sold, or equivalent merit	3
• Uses a normal approximation to find an expression for the probability, or equivalent merit	2
• Writes a probability statement similar to $P(X > 350) \leq 0.01$, where X is the number of passengers who turn up, or equivalent merit	1

Sample answer:

METHOD 1:

Let n be the number of tickets sold for a 350-seat flight ($n \geq 350$).

Let X be the number of passengers showing up on a 350-seat flight $X \sim \text{Bin}(n,\ 0.95)$.

The management's decision can be written as:
$P(X > 350) \leq 0.01$
or
$P(X \leq 350) \geq 0.99$

Find n such that $P(X \leq 350) = 0.99$.

Approximating X by the appropriate normal random variable Y, without continuity correction, the condition $P(X \leq 350) \geq 0.99$ can be rewritten.

$P(Y \leq 350) \geq 0.99$.

Need to find n such that $P(Y \geq 350) = 0.99$

This becomes

$$P\left(\frac{Y - 0.95n}{\sqrt{n \times 0.95 \times 0.05}} < \frac{350 - 0.95n}{\sqrt{n \times 0.95 \times 0.05}}\right) = 0.99$$

This happens if

$$\frac{350 - 0.95n}{\sqrt{0.95 \times 0.05n}} \simeq 2.33$$

$$350 - 0.95n = 2.33\sqrt{0.95 \times 0.05n}$$

Let $x = \sqrt{n}$

$$0.95x^2 + 2.33\sqrt{0.95 \times 0.05}x - 350 = 0$$

$$x = \sqrt{n} = \frac{-2.33\sqrt{0.95 \times 0.05} \pm \sqrt{(2.33)^2 \times 0.95 \times 0.05 + 4 \times 0.95 \times 350}}{2 \times 0.95}$$

$\sqrt{n} \approx 18.93$

$n \simeq 358.30$

So $n = 358$ (359 would result in getting too many passengers more than 1% of the time)

Given the management's decision, the optimal number of tickets sold on a 350-seat flight is 358.

Answers could include:

METHOD 2: (with continuity correction)

X, Y and n defined as before.

Approximate the binomial X by the appropriate normal random variable Y, with continuity correction.

$P(Y < 350.5) \geq 0.99$ (i)

$E(Y) = E(X) = n \times 0.95$

$\text{Var}(Y) = n \times 0.95 \times 0.05$

(i) becomes

$$P\left(\frac{Y - 0.95n}{\sqrt{n \times 0.95 \times 0.05}} < \frac{350.5 - 0.95n}{\sqrt{n \times 0.95 \times 0.05}}\right) = 0.99$$

Using the normal distribution table

$$\frac{350.5 - 0.95n}{\sqrt{0.95 \times 0.05n}} \simeq 2.33$$

$$350.5 - 0.95n = 2.33\sqrt{0.95 \times 0.05} \times \sqrt{n}$$

Let $x = \sqrt{n}$

$$0.95x^2 + 2.33\sqrt{0.95 \times 0.05}x - 350.5 = 0$$

$$\Delta = b^2 - 4ac = 0.95 \times 0.05 \times (2.33)^2 - 4 \times 0.95 \times (-350.5)$$

$$\simeq 1332.158$$

$$x = \sqrt{n} = \frac{-2.33\sqrt{0.95 \times 0.05} + \sqrt{1332.158}}{2 \times 0.95}$$

(Can ignore the negative solution since $x = \sqrt{n} > 0$.)

$\sqrt{n} \approx 18.94$

$n \simeq 358.8$

$n = 358$ (359 would result in getting too many passengers more than 1% of the time)

2022 HSC Mathematics Extension 1 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	ME-T1 Inverse Trigonometric Functions	ME11-3
2	1	ME-F1 Further Work with Functions	ME11-1
3	1	ME-F2 Polynomials	ME11-2
4	1	ME-F1 Further Work with Functions	ME11-1
5	1	ME-F1 Further Work with Functions	ME11-2
6	1	ME-V1 Introduction to Vectors	ME12-2
7	1	ME-A1 Working with Combinatorics	ME11-5
8	1	ME-V1 Introduction to Vectors	ME12-2
9	1	ME-C2 Further Calculus Skills	ME12-1
10	1	ME-C3 Applications of Calculus	ME12-4

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	ME-V1 Introduction to Vectors	ME12-2
11 (a) (ii)	1	ME-V1 Introduction to Vectors	ME12-2
11 (b)	3	ME-C2 Further Calculus Skills	ME12-1
11 (c)	2	ME-A1 Working with Combinatorics	ME11-5
11 (d)	2	ME-V1 Introduction to Vectors	ME12-2
11 (e)	3	ME-T3 Trigonometric Equations	ME12-3
11 (f)	3	ME-F1 Further Work with Functions	ME11-1
12 (a)	2	ME-C3 Applications of Calculus	ME12-1
12 (b)	2	ME-A1 Working with Combinatorics	ME11-5
12 (c)	3	ME-C2 Further Calculus Skills	ME12-1
12 (d) (i)	3	ME-C3 Applications of Calculus	ME11-4, ME12-4
12 (d) (ii)	1	ME-C1 Rate of Change	ME11-4
12 (e)	2	ME-S1 The Binomial Distribution	ME12-5
12 (f)	3	ME-P1 Proof by Mathematical Induction	ME12-1
13 (a)	3	ME-V1 Introduction to Vectors	ME12-2

Question	Marks	Content	Syllabus outcomes
13 (b)	3	ME-C3 Applications of Calculus	ME12-4
13 (c)	2	ME-T1 Inverse Trigonometric Functions	ME11-1
13 (d)	3	ME-F2 Polynomials	ME11-1
13 (e) (i)	2	ME-S1 The Binomial Distribution	ME12-5, ME12-7
13 (e) (ii)	1	ME-S1 The Binomial Distribution	ME12-5, ME12-7
14 (a)	4	ME-C3 Applications of Calculus	ME12-4
14 (b)	3	ME-V1 Introduction to Vectors	ME12-2, ME12-7
14 (c)	4	ME-V1 Introduction to Vectors ME-C3 Applications of Calculus	ME12-2, ME12-3
14 (d)	4	ME-S1 The Binomial Distribution ME-S1 The Binomial Distribution	ME12-5, ME12-7