NSW EDUCATION STANDARDS AUTHORITY

# 2017 HSC Mathematics Extension 1
# Marking Guidelines

## Section I

### Multiple-choice Answer Key

| Question | Answer |
|---|---|
| 1 | A |
| 2 | B |
| 3 | B |
| 4 | C |
| 5 | D |
| 6 | D |
| 7 | A |
| 8 | C |
| 9 | C |
| 10 | B |

# Section II

## Question 11 (a)

| Criteria | Marks |
|---|---|
| • Provides correct answer | 1 |

***Sample answer:***

$$x \text{ value of } P = \frac{2 \times 1 + 3 \times -4}{2+3}$$
$$= -\frac{10}{5}$$
$$= -2$$

## Question 11 (b)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Attempts to use the chain rule, or equivalent merit | 1 |

***Sample answer:***

Let $y = \tan^{-1}\left(x^3\right)$

$$y' = \frac{1}{1+\left(x^3\right)^2} \times 3x^2$$
$$= \frac{3x^2}{1+x^6}$$

## Question 11 (c)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 3 |
| • Correctly identifies 1 and –1 as important, or equivalent merit | 2 |
| • Attempts to deal with the denominator, or equivalent merit | 1 |

***Sample answer:***

$$\frac{2x}{x+1} > 1$$

Multiply both sides by $(x + 1)^2$

$$2x(x+1) > (x+1)^2$$

$$2x^2 + 2x > x^2 + 2x + 1$$

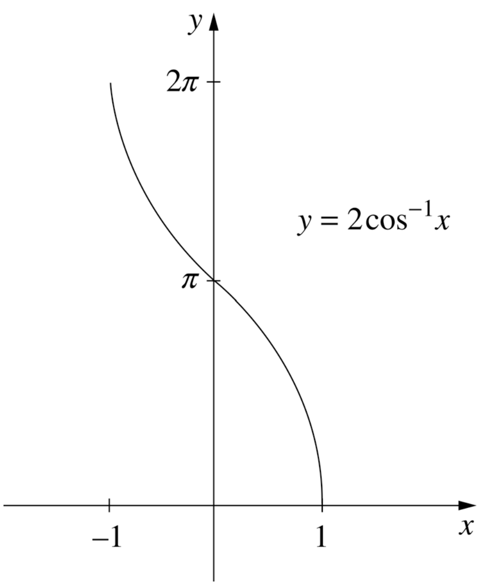
$$x^2 > 1$$

$$\therefore\ x > 1 \quad \text{or} \quad x < -1$$

## Question 11 (d)

| Criteria | Marks |
|---|---|
| • Provides correct sketch | 2 |
| • Indicates correct range, or equivalent merit | 1 |

***Sample answer:***



### Question 11 (e)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 3 |
| • Provides a correct primitive, or equivalent merit | 2 |
| • Attempts to use given substitution, or equivalent merit | 1 |

***Sample answer:***

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx \qquad x = u^2 - 1$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u\,du$$

when $x = 0$ $\quad u^2 - 1 = 0 \quad u = 1$
when $x = 3$ $\quad u^2 - 1 = 3 \quad u = 2$

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u^2-1}{u}.2u\,du$$

$$= 2\int_1^2 (u^2-1)\,du$$

$$= 2\left[\frac{u^3}{3} - u\right]_1^2$$

$$= 2\left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right]$$

$$= \frac{8}{3}$$

### Question 11 (f)

| Criteria | Marks |
|---|---|
| • Provides correct primitive | 1 |

***Sample answer:***

$$\int \sin^2 x \cos x\,dx$$

$$= \frac{\sin^3 x}{3} + c$$

## Question 11 (g) (i)

| Criteria | Marks |
|---|---|
| • Provides correct answer | 1 |

***Sample answer:***

$$\binom{8}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^5$$

## Question 11 (g) (ii)

| Criteria | Marks |
|---|---|
| • Provides correct answer | 1 |

***Sample answer:***

$$\left(\frac{4}{5}\right)^8$$

## Question 11 (g) (iii)

| Criteria | Marks |
|---|---|
| • Provides correct answer | 1 |

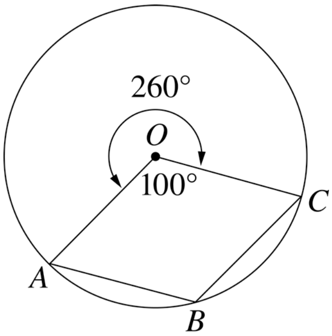
***Sample answer:***

$$1-\left(\frac{4}{5}\right)^8$$

**Question 12 (a)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Attempts to relate two angles in the diagram, or equivalent merit | 1 |

***Sample answer:***




Reflex $\angle AOC = 360° - 100°$ (angle of revolution)

$= 260°$

$\angle ABC = \frac{1}{2}$ reflex $\angle AOC$ (angle at centre equals twice the angle at circumference standing in the same arc)

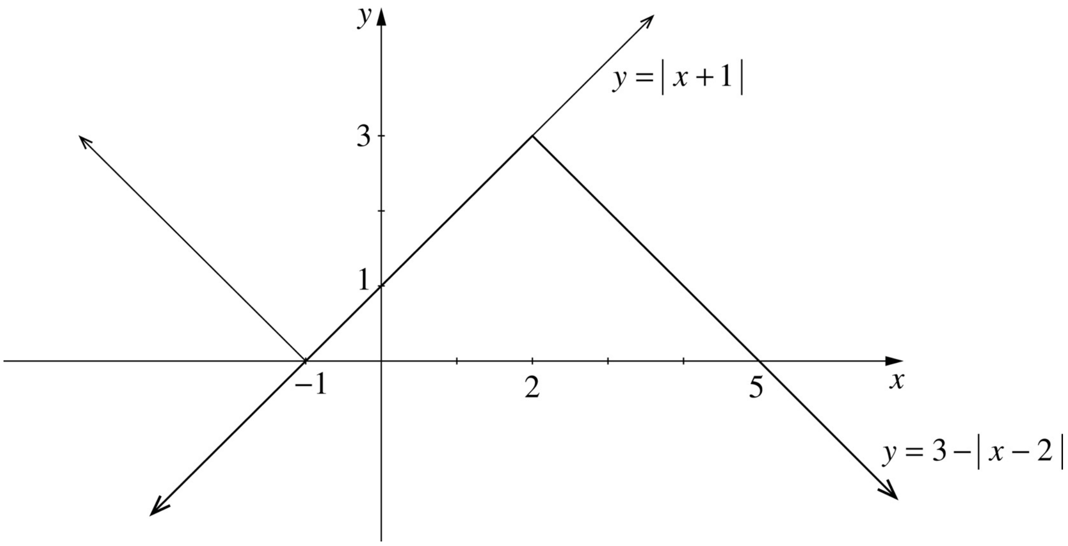
$= \frac{1}{2}(260°)$

$= 130°$

## Question 12 (b) (i)

| Criteria | Marks |
|---|---|
| • Provides correct sketch | 3 |
| • Provides correct sketch of $y = 3 - \lvert x - 2 \rvert$, or equivalent merit | 2 |
| • Provides correct sketch of $y = \lvert x + 1 \rvert$, or equivalent merit | 1 |

***Sample answer:***




## Question 12 (b) (ii)

| Criteria | Marks |
|---|---|
| • Provides correct answer | 1 |

***Sample answer:***

$\therefore\ -1 \le x \le 2$

when $\lvert x+1 \rvert = 3 - \lvert x-2 \rvert$ the graphs have points in common.

ie $\lvert x+1 \rvert + \lvert x-2 \rvert = 3$

**Question 12 (c) (i)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 3 |
| • Equates the volumes in the correct ratio and attempts to evaluate an integral, or equivalent merit | 2 |
| • Provides correct integral for the volume of one solid, or equivalent merit | 1 |

***Sample answer:***

$$V_1 = \pi \int_h^1 y^2 \, dx$$

$$= \pi \int_h^1 1 - x^2 \, dx$$

$$= \pi \left[ x - \frac{x^3}{3} \right]_h^1$$

$$= \pi \left[ 1 - \frac{1}{3} \right] - \pi \left[ h - \frac{h^3}{3} \right]$$

$$= \pi \left[ \frac{2}{3} - h + \frac{h^3}{3} \right]$$

$$V_2 = \pi \int_{-1}^h 1 - x^2 \, dx$$

$$= \pi \left[ x - \frac{x^3}{3} \right]_{-1}^h$$

$$= \pi \left[ h - \frac{h^3}{3} \right] - \pi \left[ -1 - \frac{-1}{3} \right]$$

$$= \pi \left[ h - \frac{h^3}{3} + \frac{2}{3} \right]$$

Ratio $V_2 : V_1 = 2 : 1 \qquad \therefore V_2 = 2V_1$

$$h - \frac{h^3}{3} + \frac{2}{3} = 2\left( \frac{2}{3} - h + \frac{h^3}{3} \right)$$

$$= \frac{4}{3} - 2h + \frac{2h^3}{3}$$

$\therefore \quad 3h - h^3 + 2 = 4 - 6h + 2h^3$

$\therefore \; 3h^3 - 9h + 2 = 0$

**Question 12 (c) (ii)**

| Criteria | Marks |
|---|---|
| • Provides correct answer | 1 |

***Sample answer:***

Let $f(h) = 3h^3 - 9h + 2 \qquad h_1 = 0$

$$h_2 = h_1 - \frac{f(h_1)}{f'(h_1)} = 0 - \frac{f(0)}{f'(0)}$$

$$f(h_1) = f(0) = 2$$

$$f'(h) = 9h^2 - 9$$

$$\therefore \; f'(0) = 9(0)^2 - 9 = -9$$

$$\therefore \quad h_2 = 0 - \frac{2}{-9} = \frac{2}{9}$$

## Question 12 (d)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 3 |
| • Attempts to use $\ddot{x} = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$, or equivalent merit | 2 |
| • Correctly finds $\dfrac{dt}{dx}$, or equivalent merit | 1 |

***Sample answer:***

$$t = 4 - e^{-2x}$$

$$\frac{dt}{dx} = +2e^{-2x}$$

$$\therefore \frac{dx}{dt} = \frac{1}{2}e^{2x}$$

$$v = \frac{1}{2}e^{2x}$$

$$v^2 = \frac{1}{4}e^{4x}$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{8}e^{4x}\right)$$

$$= \frac{4}{8}e^{4x}$$

$$\ddot{x} = \frac{1}{2}e^{4x}$$

## Question 12 (e)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Correctly uses double angle result, or equivalent merit | 1 |

***Sample answer:***

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \text{ since } \cos 2x = 1 - 2\sin^2 x$$

$$= 2 \lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

$$= 2 \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^2$$

$$= 2 \times 1^2$$

$$= 2$$

## Question 13 (a)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 3 |
| • Correctly finds the value of $n^2$, or equivalent merit | 2 |
| • Correctly uses $v^2 = n^2\left(a^2 - x^2\right)$, or equivalent merit | 1 |

***Sample answer:***

$$v^2 = n^2\left(a^2 - x^2\right)$$

$\therefore \quad 4^2 = n^2a^2 - 2^2n^2$ ——— ①

and $\quad 3^2 = n^2a^2 - 5^2n^2$ ——— ②

①–② $\quad \therefore 7 = -4n^2 + 25n^2$

$$21n^2 = 7$$

$$n^2 = \frac{1}{3}$$

$$n = \frac{1}{\sqrt{3}}, \quad n > 0$$

Period, $T = \dfrac{2\pi}{n}$

$$= \frac{2\pi}{\frac{1}{\sqrt{3}}}$$

$$= 2\pi\sqrt{3}$$

## Question 13 (b) (i)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Correctly applies binomial theorem to one binomial expression, or equivalent merit | 1 |

***Sample answer:***

$n$ is even

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n$$

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + (-1)^r\binom{n}{r}x^2 + \cdots + \binom{n}{n}x^n$$

$$(1-x)^n + (1-x)^n = 2\binom{n}{0} + 2\binom{n}{2}x^2 + \cdots + 2\binom{n}{n}x^n$$

(odd terms have different sign and so cancel)

$$(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \cdots + \binom{n}{n}x^n\right]$$

## Question 13 (b) (ii)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 1 |

***Sample answer:***

Differentiating with respect to $x$.

$$n(1+x)^{n-1} - n(1-x)^{n-1} = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right]$$

$$n\left((1+x)^{n-1} - (1-x)^{n-1}\right) = 2\left[2\binom{n}{2}x + 4\binom{n}{4}x^3 + \cdots + n\binom{n}{n}x^{n-1}\right]$$

**Question 13 (b) (iii)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Uses $x = 1$, or equivalent merit | 1 |

***Sample answer:***

Let $x = 1$

$$n\left(2^{n-1} - 0\right) = 2\left[2\binom{n}{2} + 4\binom{n}{4} + \cdots + n\binom{n}{n}\right]$$

$$n\,2^{n-2} = 2\binom{n}{2} + 4\binom{n}{4} + \cdots + n\binom{n}{n}$$

$$n\,2^{n-3} = \binom{n}{2} + 2\binom{n}{4} + 3\binom{n}{6} + \cdots + \frac{n}{2}\binom{n}{n}$$

**Question 13 (c) (i)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Attempts to solve $y = 0$, or equivalent merit | 1 |

***Sample answer:***

Set $0 = y = Vt\sin\theta - \frac{1}{2}gt^2$

$$t = 0 \quad \text{or} \quad V\sin\theta = \frac{1}{2}gt$$

$$t = \frac{2V\sin\theta}{g}$$

At this value

$$x = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta$$

$$= \frac{V^2}{g}\sin 2\theta$$

**Question 13 (c) (ii)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 1 |

***Sample answer:***

$$\text{Range} = \frac{V^2}{g}\sin 2\theta$$

If $v^2 < 100g$, then range $\frac{100g\sin 2\theta}{g}$

ie range $< 100\sin 2\theta$

since $\sin 2\theta \le 1$

then range $< 100$

**Question 13 (c) (iii)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Provides an inequality in $\sin 2\theta$, or equivalent merit | 1 |

***Sample answer:***

If $V^2 = 200g$, then range $= \frac{200g\sin 2\theta}{g} \ge 100$

ie $200\sin 2\theta \ge 100$

$$\sin 2\theta \ge \frac{1}{2}$$

Now, $0 \le \theta \le \frac{\pi}{2}$ or $0 \le 2\theta \le \pi$

$$\frac{\pi}{6} \le 2\theta \le \frac{5\pi}{6}$$

$$\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$$

**Question 13 (c) (iv)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Obtains correct expression for the maximum height, or equivalent merit | 1 |

***Sample answer:***

Maximum height occurs when

$$0 = \frac{dy}{dt} = V\sin\theta - gt$$

$$t = \frac{V}{g}\sin\theta$$

At this time

$$y = \frac{V^2}{g}\sin^2\theta - \frac{1}{2}g\frac{V^2}{g^2}\sin^2\theta$$

$$= \frac{V^2}{g}\frac{\sin^2\theta}{2}$$

$$= 100\sin^2\theta \qquad \left(\frac{V^2}{g} = 200\right)$$

For $0 \le \theta \le \frac{\pi}{2}$ $\sin^2\theta$ is an increasing function,

so maximum occurs when $\theta = \frac{5\pi}{12}$

$\therefore$ greatest height is $100\sin^2\left(\frac{5\pi}{12}\right)$

## Question 14 (a)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 3 |
| • Correctly shows $P(n) \Rightarrow P(n+1)$, or equivalent merit | 2 |
| • Establishes result for $n = 1$, or equivalent merit | 1 |

***Sample answer:***

Let $P(n)$ be the given proposition.

$P(1)$ is true since $8^3 + 6 = 518 = 7 \times 74$ which is divisible by 7.

Let $k$ be an integer for which $P(k)$ is true.

That is $8^{2k+1} + 6^{2k-1} = 7m$, for some integer $m$.

Consider $8^{2(k+1)+1} + 6^{2(k+1)-1}$

$= 8^2 \times 8^{2k+1} + 6^2 \times 6^{2k-1}$

$= 8^2 \times 8^{2k+1} + 8^2 \times 6^{2k-1} - 8^2 \times 6^{2k-1} + 6^2 \times 6^{2k-1}$

$= 8^2\left(8^{2k+1} + 6^{2k-1}\right) + \left(6^2 - 8^2\right)6^{2k-1}$

$= 64 \times 7m - 28 \times 6^{2k-1}$

$= 7\left(64m - 4 \times 6^{2k-1}\right)$

which is divisible by 7, since $m$ and $k$ are integers and $64m - 4 \times 6^{2k-1}$ is an integer.

$\therefore$ $P(k + 1)$ is also true

$\therefore$ $P(n)$ is true for all $n \geq 1$ by induction.

**Question 14 (b) (i)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Obtains equation of tangent at $P$, or equivalent merit | 1 |

***Sample answer:***

The equation of the tangent at $P$ is

$$y = px - p^2$$

substituting into $x^2 = -4ay$

$$-\frac{x^2}{4a} = px - p^2$$

$\therefore\ x^2 + 4apx - 4ap^2 = 0$

**Question 14 (b) (ii)**

| Criteria | Marks |
|---|---|
| • Provides correct coordinates | 2 |
| • Correctly verifies one coordinate | 1 |

***Sample answer:***

The sum of the roots of the equation is $-4ap$.
$x$-coordinate of $M$ is average of the roots
$\therefore$ $x$-coordinate of $M$ is $-2ap$.

Substituting into the tangent,

$y = p \times -2ap - p^2$

$= -p^2(2a+1)$

$\therefore M$ is the point $\left(-2ap, -p^2(2a+1)\right)$

### Question 14 (b) (iii)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Eliminates the parameter, or equivalent merit | 1 |

***Sample answer:***

Let $x = -2ap$, $y = -p^2(2a+1)$

$$\therefore y = -\left(\frac{x}{2a}\right)^2\left(2a+1\right)$$

$$= -x^2\frac{(2a+1)}{4a^2}$$

so $$x^2 = -\frac{4a^2}{2a+1}y$$

Hence $\dfrac{a^2}{2a+1} = 1$

ie $a^2 - 2a - 1 = 0$

$$\therefore a = 1 \pm \sqrt{2}$$

$$\therefore a = 1 + \sqrt{2}, \text{ since } a > 0$$

### Question 14 (c) (i)

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Correctly applies the product rule, or equivalent merit | 1 |

***Sample answer:***

$$\frac{d}{dt}\left(F(t)e^{0.4t}\right) = F'(t)e^{0.4t} + 0.4F(t)e^{0.4t}$$

$$= e^{0.4t}\left(50e^{-0.5t} - 0.4F(t)\right) + 0.4F(t)e^{0.4t}$$

$$= 50e^{-0.1t}$$

## Question 14 (c) (ii)

| Criteria | Marks |
| --- | --- |
| • Provides correct solution | 2 |
| • Correctly integrates the expression given in part (i), or equivalent merit | 1 |

***Sample answer:***

Integrating,

$$F(t)e^{0.4t} = \int 50e^{-0.1t}\,dt$$

$$= -500e^{-0.1t} + c$$

Now $F(t) = 0$ when $t = 0$ $\therefore$ $c = +500$

$$\therefore \quad F(t)e^{0.4t} = 500\left(1 - e^{-0.1t}\right)$$

$$F(t) = 500\left(e^{-0.4t} - e^{-0.5t}\right)$$

**Question 14 (c) (iii)**

| Criteria | Marks |
|---|---|
| • Provides correct solution | 2 |
| • Makes substantial progress | 1 |

***Sample answer:***

$$F'(t) = 500\left(-0.4e^{-0.4t} + 0.5e^{-0.5t}\right)$$

$$= 0 \text{ for a maximum}$$

$$\therefore 0.4e^{-0.4t} = 0.5e^{-0.5t}$$

$$\therefore \frac{5}{4} = e^{0.1t}$$

$$t = 10\ln\frac{5}{4}$$

$$(\simeq 2.23 \text{ hours})$$

<u>Alternative</u>

$$F'(t) = 50e^{-0.5t} - 0.4F(t)$$

$$= 50e^{-0.5t} - 0.4\left[500\left(e^{-0.4t} - e^{-0.5t}\right)\right]$$

$$= 250e^{-0.5t} - 200e^{-0.4t}$$

For maximum $F'(t) = 0$

$$\therefore 250e^{-0.5t} = 200e^{-0.4t}$$

$$\frac{5}{4} = \frac{e^{-0.4t}}{e^{-0.5t}}$$

$$= e^{0.1t}$$

$$\therefore 0.1t = \ln\left(\frac{5}{4}\right)$$

$$t = 10\ln\left(\frac{5}{4}\right)$$

# 2017 HSC Mathematics Extension 1 Mapping Grid

**Section I**

| Question | Marks | Content | Syllabus outcomes |
|---|---|---|---|
| 1 | 1 | 16.2 | PE3 |
| 2 | 1 | 12.1 | HE7 |
| 3 | 1 | 2.10 | PE3 |
| 4 | 1 | 5.9 | PE2 |
| 5 | 1 | 10.5 | HE7 |
| 6 | 1 | 9.6 | PE3 |
| 7 | 1 | 15.3 | HE4 |
| 8 | 1 | 14.1 | HE4 |
| 9 | 1 | 17.3 | HE3 |
| 10 | 1 | 18.1 | HE3 |

**Section II**

| Question | Marks | Content | Syllabus outcomes |
|---|---|---|---|
| 11 (a) | 1 | 6.7 | PE6 |
| 11 (b) | 2 | 8.8, 15.5 | HE4 |
| 11 (c) | 3 | 1.4 | PE3 |
| 11 (d) | 2 | 15.3 | HE4 |
| 11 (e) | 3 | 11.5 | HE6 |
| 11 (f) | 1 | 13.6E | HE6 |
| 11 (g) (i) | 1 | 18.2 | HE3 |
| 11 (g) (ii) | 1 | 18.2 | HE3 |
| 11 (g) (iii) | 1 | 18.2 | HE3 |
| 12 (a) | 2 | 2.8 | PE3 |
| 12 (b) (i) | 3 | 1.4, 1.2 | PE6 |
| 12 (b) (ii) | 1 | 1.4, 1.2 | PE2 |
| 12 (c) (i) | 3 | 11.4 | H8, HE7 |
| 12 (c) (ii) | 1 | 16.4 | HE7 |
| 12 (d) | 3 | 14.3 | HE5 |
| 12 (e) | 2 | 5.7, 8.2, 13.4E | HE7 |
| 13 (a) | 3 | 14.4 | HE3 |
| 13 (b) (i) | 2 | 17.1, 17.3 | HE7 |
| 13 (b) (ii) | 1 | 17.1, 17.3 | HE7 |
| 13 (b) (iii) | 2 | 17.1, 17.3 | HE7 |
| 13 (c) (i) | 2 | 14.3 | HE3 |
| 13 (c) (ii) | 1 | 14.3 | HE3 |
| 13 (c) (iii) | 2 | 14.3 | HE3 |

| Question | Marks | Content | Syllabus outcomes |
|---|---|---|---|
| 13 (c) (iv) | 2 | 14.3 | HE3 |
| 14 (a) | 3 | 7.4 | HE2 |
| 14 (b) (i) | 2 | 9.6 | PE3, PE4 |
| 14 (b) (ii) | 2 | 9.6 | PE3, PE4 |
| 14 (b) (iii) | 2 | 9.6 | PE3, PE4 |
| 14 (c) (i) | 2 | 8.8, 11.2, 12.5 | HE3 |
| 14 (c) (ii) | 2 | 12.5 | HE3 |
| 14 (c) (iii) | 2 | 10.6, 12.5 | HE3 |