



NSW Education Standards Authority

IMPORTANT:

This resource was developed to support a previous version of the syllabus. It may contain content that differs from the current syllabus.

2019 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

**General
Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

**Total marks:
100**

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–17)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

- 1 What is the value of $(3 - 2i)^2$?
- A. $5 - 12i$
 - B. $5 + 12i$
 - C. $13 - 12i$
 - D. $13 + 12i$
- 2 Which of the following is a primitive of $\frac{\sin x}{\cos^3 x}$?
- A. $\frac{1}{2}\sec^2 x$
 - B. $-\frac{1}{2}\sec^2 x$
 - C. $\frac{1}{4}\sec^4 x$
 - D. $-\frac{1}{4}\sec^4 x$
- 3 Which expression is equal to $\int x \cos x \, dx$?
- A. $-x \sin x + \cos x + C$
 - B. $-x \sin x - \cos x + C$
 - C. $x \sin x + \cos x + C$
 - D. $x \sin x - \cos x + C$

- 4 The polynomial $2x^3 + bx^2 + cx + d$ has roots 1 and -3 , with one of them being a double root.

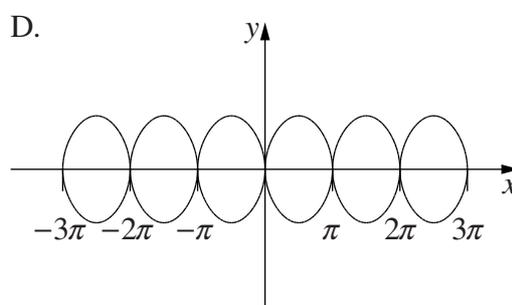
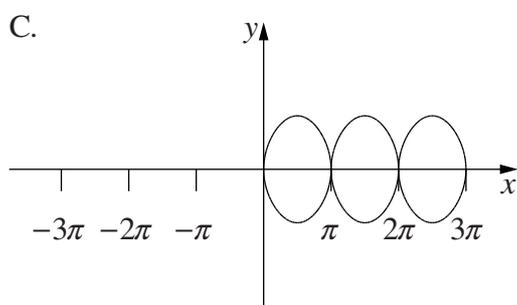
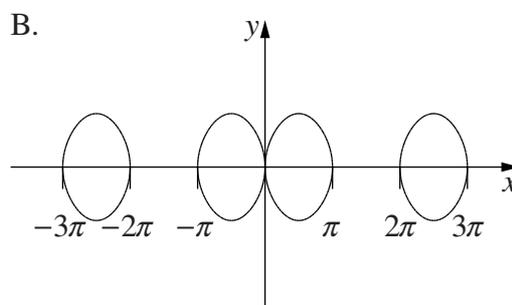
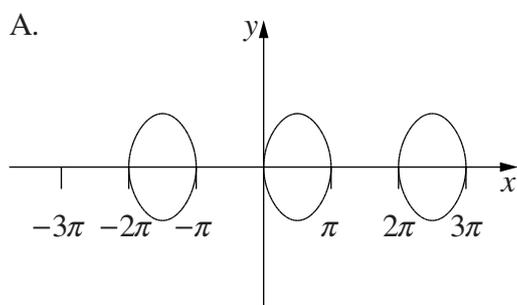
What is a possible value of b ?

- A. -10
 B. -5
 C. 5
 D. 10
- 5 Which of the following best describes the locus of points in the Argand plane given by

$$|z - 2| + |z + 1| = 7?$$

- A. An ellipse with foci at $(-3, 0)$ and $(4, 0)$
 B. An ellipse with foci at $(-1, 0)$ and $(2, 0)$
 C. A hyperbola with foci at $(-3, 0)$ and $(4, 0)$
 D. A hyperbola with foci at $(-1, 0)$ and $(2, 0)$

- 6 Which graph best represents $y^2 = 2 \sin|x|$?



7 Which of these integrals has the largest value?

A. $\int_0^{\frac{\pi}{4}} \tan x \, dx$

B. $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

C. $\int_0^{\frac{\pi}{4}} 1 - \tan x \, dx$

D. $\int_0^{\frac{\pi}{4}} 1 - \tan^2 x \, dx$

8 Let z be a complex number such that $z^2 = -i\bar{z}$.

Which of the following is a possible value for z ?

A. $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

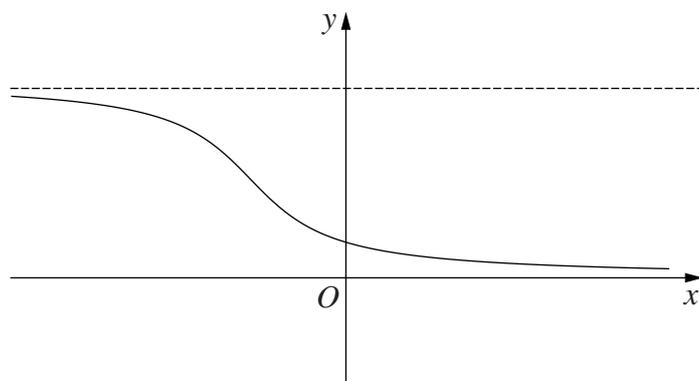
B. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

C. $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

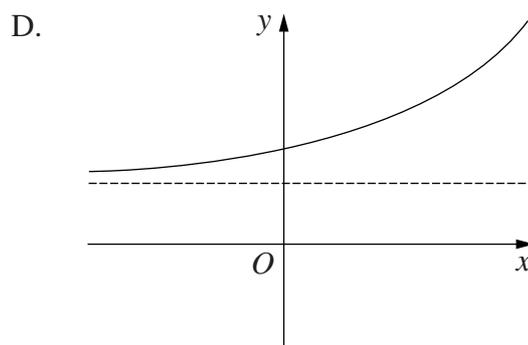
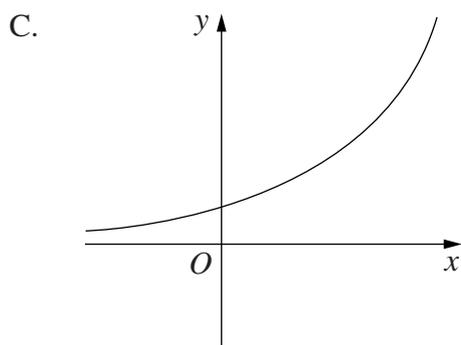
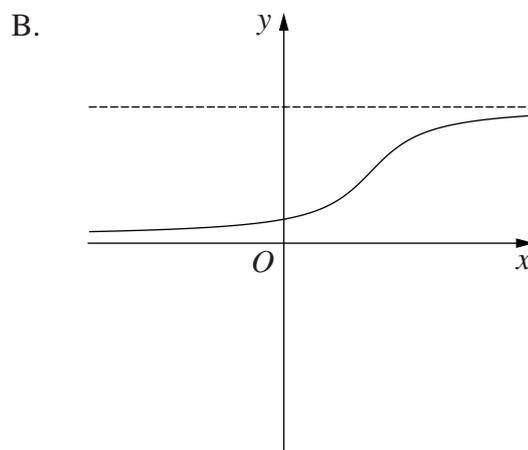
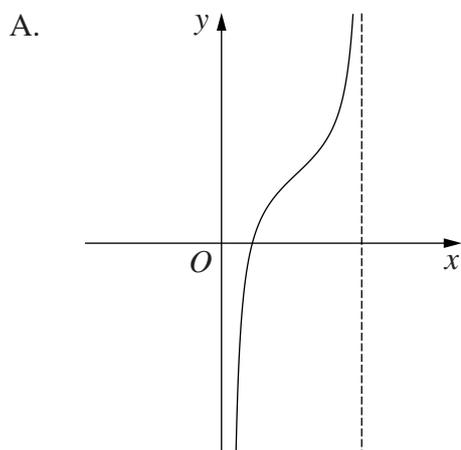
D. $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

- 9 It is given that the function $f(x)$ has an inverse. Let $g(x) = f^{-1}(x)$.

The diagram shows the graph of the derivative $y = f'(x)$.



Which graph best represents the graph of $y = g'(x)$?



- 10** An access code consists of 4 digits chosen from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The code will only work if the digits are entered in the correct order.

Some access codes contain exactly two different digits, for example 3377 or 5155.

How many such access codes can be made using exactly two different digits?

- A. 630
- B. 900
- C. 1080
- D. 2160

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

- (a) Let $z = 1 + 3i$ and $w = 2 - i$.
- (i) Find $z + \bar{w}$. 1
- (ii) Express $\frac{z}{w}$ in the form $x + iy$, where x and y are real numbers. 2
- (b) Sketch the ellipse $\frac{x^2}{4} + y^2 = 1$, showing the intercepts, foci and directrices. 3
- (c) Find $\int \frac{dx}{x^2 + 10x + 29}$. 2
- (d) Find $\int \frac{6}{x^2 - 9} dx$. 3
- (e) Let $z = -1 + i\sqrt{3}$.
- (i) Write z in modulus–argument form. 2
- (ii) Find z^3 , giving your answer in the form $x + iy$, where x and y are real numbers. 2

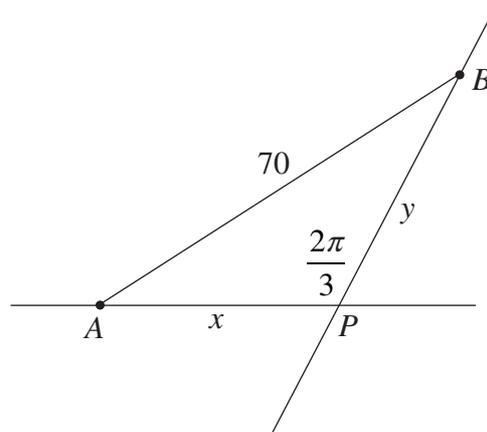
Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Sketch the region defined by $\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{2}$ and $\text{Im}(z) \leq 1$. **2**

(b) The diagram shows two straight railway tracks that meet at an angle of $\frac{2\pi}{3}$ at the point P .

Trains A and B are joined by a cable which is 70 m long.

At time t seconds, train A is x metres from P and train B is y metres from P .



Train B is towing train A and is moving at a constant speed of 4 m s^{-1} away from P .

(i) Show that $x^2 + xy + y^2 = 70^2$. **1**

(ii) What is the value of $\frac{dx}{dt}$ when train A is 30 metres from P and train B is 50 metres from P ? **3**

Question 12 continues on page 9

Question 12 (continued)

- (c) Following a volcanic eruption, fine ash is deposited in a circular pattern around the point of eruption. 4

The depth in kilometres, $D(x)$, of the fine ash can be modelled by

$$D(x) = Ae^{-kx},$$

where x is the distance in kilometres from the point of eruption and $k > 0$.

Using the method of cylindrical shells, find an expression for the volume of fine ash deposited between 10 km and 40 km from the point of eruption.

- (d) Consider the function $f(x) = x^3 - 1$.
- (i) Sketch the graph $y = |f(x)|$. 1
- (ii) Sketch the graph $y = \frac{1}{f(x)}$. 2
- (iii) Without using calculus, sketch the graph $y = \frac{x}{f(x)}$. 2

End of Question 12

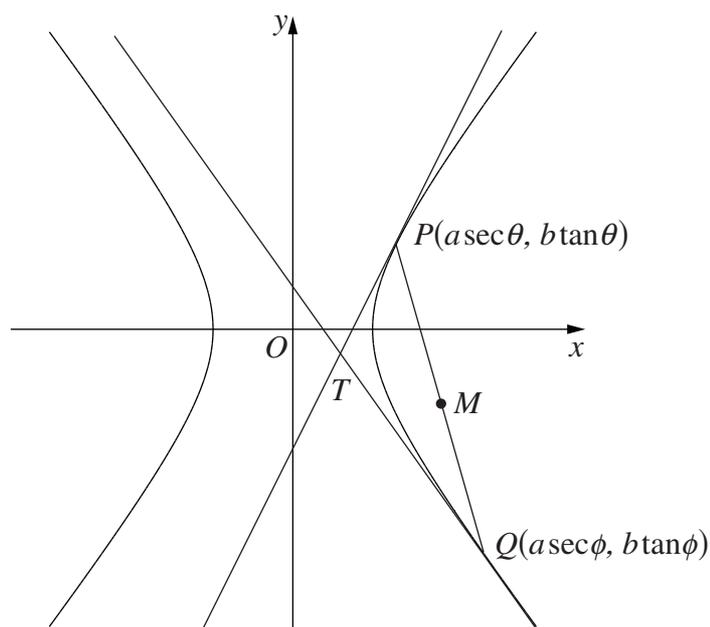
Question 13 (15 marks) Use the Question 13 Writing Booklet.

- (a) Two points $P(a \sec \theta, b \tan \theta)$ and $Q(a \sec \phi, b \tan \phi)$ are on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Let O be the origin and let M be the midpoint of PQ .

Tangents to the hyperbola are drawn at points P and Q as shown.



- (i) Show that the equation of the tangent at P is **2**

$$bx \sec \theta - ay \tan \theta = ab.$$

- (ii) The tangents at P and Q meet at the point $T(x_0, y_0)$. **2**

Show that $x_0 = \frac{a(\tan \theta - \tan \phi)}{\sec \phi \tan \theta - \sec \theta \tan \phi}$.

- (iii) It is given that $y_0 = \frac{b(\sec \theta - \sec \phi)}{\sec \phi \tan \theta - \sec \theta \tan \phi}$. (Do NOT prove this.) **3**

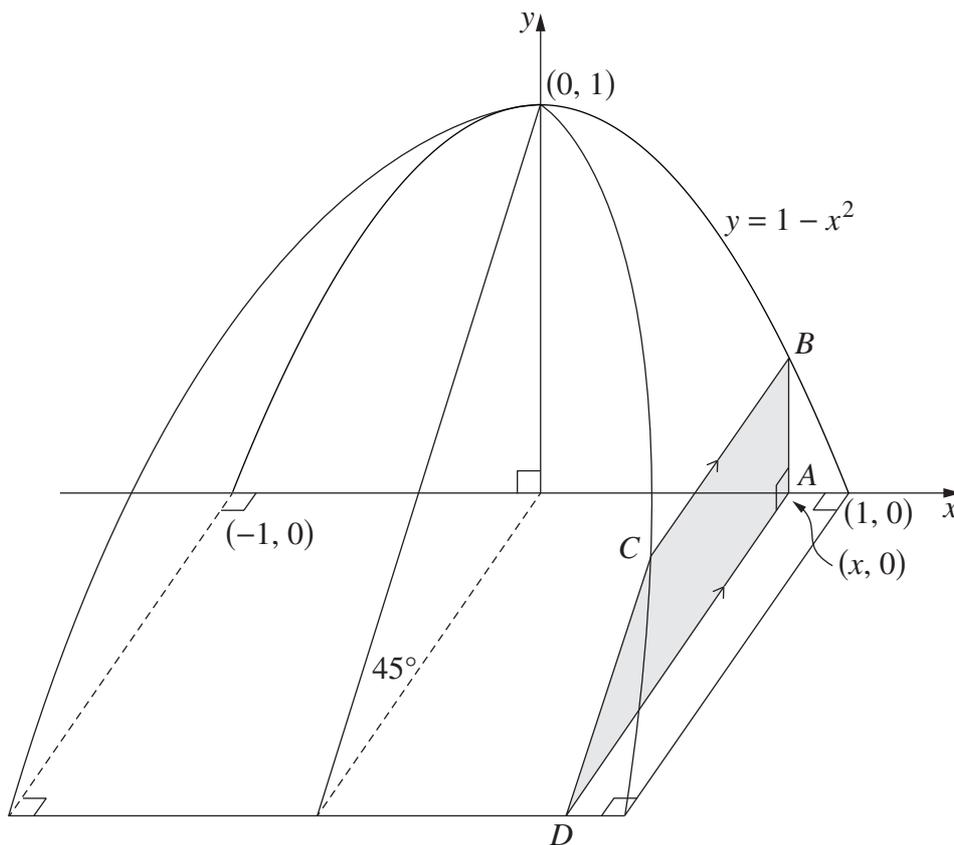
By considering gradients, or otherwise, show that O , T and M are collinear.

Question 13 continues on page 11

Question 13 (continued)

- (b) One end of a solid is the region bounded by $y = 1 - x^2$ and the x -axis. The base of the solid is a rectangle perpendicular to the x,y -plane with one side lying on the x -axis. A flat face of the solid is formed by the plane passing through the point $(0, 1)$ making an angle of 45° to the base, as shown in the diagram.

Vertical cross-sections perpendicular to the x -axis are trapezia. One trapezium $ABCD$ is shown, where A is the point $(x, 0)$.



- (i) Show that the area of $ABCD$ is $\frac{1 - x^4}{2}$. 2
- (ii) Find the volume of the solid. 1

Question 13 continues on page 12

Question 13 (continued)

- (c) Two objects are projected from the same point on a horizontal surface. Object 1 is projected with an initial velocity of 20 m s^{-1} directed at an angle of $\frac{\pi}{3}$ to the horizontal. Object 2 is projected 2 seconds later.

The equations of motion of an object projected from the origin with initial velocity v at an angle θ to the x -axis are

$$x = vt \cos \theta$$
$$y = -4.9t^2 + vt \sin \theta,$$

where t is the time after the projection of the object. Do NOT prove these equations.

- (i) Show that Object 1 will land at a distance $\frac{100\sqrt{3}}{4.9}$ m from the point of projection. **2**
- (ii) The two objects hit the horizontal plane at the same place and time. **3**

Find the initial speed and the angle of projection of Object 2, giving your answer correct to 1 decimal place.

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

- (a) A cyclist rides around a track. The track is a circle of radius r metres, banked at an angle of θ to the horizontal. **3**

Let $v \text{ m s}^{-1}$ be the velocity at which the cyclist experiences no net force parallel to the track.

By resolving forces, show that v satisfies $v^2 = gr \tan \theta$ where g is the acceleration due to gravity.

- (b) A parachutist jumps from a plane, falls freely for a short time and then opens the parachute. Let t be the time in seconds after the parachute opens, $x(t)$ be the distance in metres travelled after the parachute opens, and $v(t)$ be the velocity of the parachutist in m s^{-1} .

The acceleration of the parachutist after the parachute opens is given by

$$\ddot{x} = g - kv,$$

where $g \text{ m s}^{-2}$ is the acceleration due to gravity and k is a positive constant.

- (i) With an open parachute the parachutist has a terminal velocity of $w \text{ m s}^{-1}$. **1**

Show that $w = \frac{g}{k}$.

At the time the parachute opens, the speed of descent is $1.6w \text{ m s}^{-1}$.

- (ii) Show that it takes $\frac{1}{k} \log_e 6$ seconds to slow down to a speed of $1.1w \text{ m s}^{-1}$. **4**

- (iii) Let D be the distance the parachutist travels between opening the parachute and reaching the speed $1.1w \text{ m s}^{-1}$. **3**

Show that $D = \frac{g}{k^2} \left(\frac{1}{2} + \log_e 6 \right)$.

- (c) (i) Show that $\cot x - \cot 2x = \operatorname{cosec} 2x$. **2**

- (ii) Use mathematical induction to prove that, for all $n \geq 1$, **2**

$$\sum_{r=1}^n \operatorname{cosec}(2^r x) = \cot x - \cot(2^n x).$$

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) (i) Show that $\int_{-a}^a \frac{f(x)}{f(x) + f(-x)} dx = \int_{-a}^a \frac{f(-x)}{f(x) + f(-x)} dx.$ **2**

(ii) Hence, or otherwise, evaluate $\int_{-1}^1 \frac{e^x}{e^x + e^{-x}} dx.$ **2**

(b) An urn contains w white marbles and y yellow marbles. Two players, A and B , play a game in which they take turns to randomly draw a marble from the urn.

If player A draws a white marble then player A wins. If player B draws a yellow marble then player B wins.

Each player replaces the marble if their draw is unsuccessful. The game stops as soon as one player wins.

Player A draws first.

(i) Explain why, in the case where $w = y$, player A has a greater chance of winning than player $B.$ **1**

(ii) Show that for player B to have a greater chance of winning than player $A,$ **3**

$$\frac{y}{w} > \frac{1 + \sqrt{5}}{2}.$$

(c) (i) Show that $\int_0^1 \frac{x}{(x+1)^2} dx = \ln 2 - \frac{1}{2}.$ **2**

(ii) Let $I_n = \int_0^1 \frac{x^n}{(x+1)^2} dx.$ **3**

Show that $I_n = \frac{1}{2(n-1)} - \frac{n}{n-1} I_{n-1}$ for $n \geq 2.$

(iii) Evaluate $I_3.$ **2**

Question 16 (15 marks) Use the Question 16 Writing Booklet.

- (a) (i) Consider the equation $x^3 - px + q = 0$, where p and q are real numbers and $p > 0$. **2**

$$\text{Let } r = \sqrt{\frac{4p}{3}} \text{ and } \cos 3\theta = \frac{-4q}{r^3}.$$

Show that $r \cos \theta$ is a root of $x^3 - px + q = 0$.

You may use the result $4 \cos^3 \theta - 3 \cos \theta = \cos 3\theta$. (Do NOT prove this.)

- (ii) Let α , β and γ be the roots of the equation **2**

$$x^3 + 9x^2 + 15x - 17 = 0.$$

Show that $\alpha + 3$, $\beta + 3$, $\gamma + 3$ are the roots of $x^3 - 12x - 8 = 0$.

- (iii) Using parts (i) and (ii), solve **3**

$$x^3 + 9x^2 + 15x - 17 = 0.$$

Question 16 continues on page 16

Question 16 (continued)

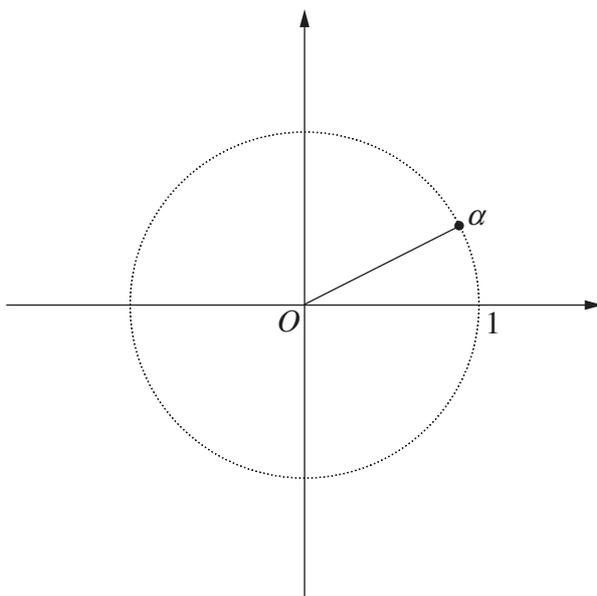
(b) Let $P(z) = z^4 - 2kz^3 + 2k^2z^2 + mz + 1$, where k and m are real numbers.

The roots of $P(z)$ are $\alpha, \bar{\alpha}, \beta, \bar{\beta}$.

It is given that $|\alpha| = 1$ and $|\beta| = 1$.

(i) Show that $(\operatorname{Re}(\alpha))^2 + (\operatorname{Re}(\beta))^2 = 1$. **3**

(ii) The diagram shows the position of α . **2**



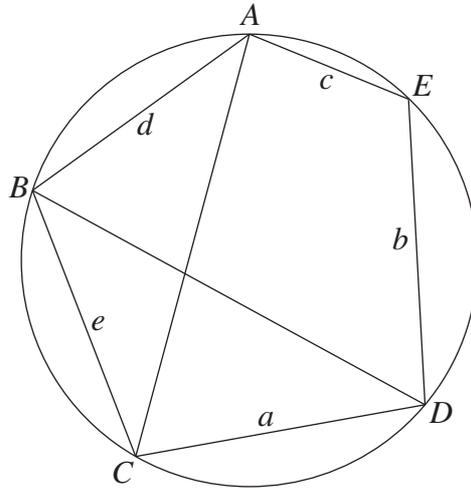
Copy or trace the diagram into your writing booklet.

On the diagram, accurately show all possible positions of β .

Question 16 continues on page 17

Question 16 (continued)

- (c) The vertices A, B, C, D and E of a pentagon lie on a circle. The lengths of the sides opposite A, B, C, D and E are a, b, c, d and e , respectively, as shown. **3**



Copy or trace the diagram into your writing booklet.

Using the same convention as in the sine rule, the vertex name also represents the size of the angle at that vertex. For example, A is used to represent the size of $\angle BAE$.

Show that
$$\frac{a}{\sin(B + E)} = \frac{d}{\sin(C + E)}.$$

End of paper

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2019 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

- Mathematics –
- Mathematics Extension 1 –
- Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n - 2) \times 180^\circ$$

Equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

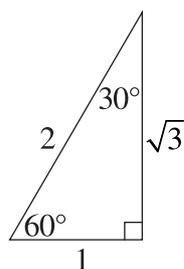
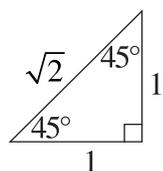
$$\sec \theta = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2} ab \sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

n th term of an arithmetic series

$$T_n = a + (n - 1)d$$

Sum to n terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n - 1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

n th term of a geometric series

$$T_n = ar^{n-1}$$

Sum to n terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P \left(1 + \frac{r}{100} \right)^n$$

Mathematics (continued)

Differentiation from first principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u) \frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x) \cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x) \sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x) \sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x - h)^2 = \pm 4a(y - k)$$

Integrals

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

$$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C$$

$$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$$

$$\int \sec^2(ax + b) dx = \frac{1}{a} \tan(ax + b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x) dx \approx \frac{b-a}{2} [f(a) + f(b)]$$

Simpson's rule (one application)

$$\int_a^b f(x) dx \approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$$180^\circ = \pi \text{ radians}$$

Length of an arc

$$l = r\theta$$

Area of a sector

$$\text{Area} = \frac{1}{2} r^2 \theta$$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \tan\phi}$$

t formulae

If $t = \tan \frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$$\sin\theta = a, \quad \theta = n\pi + (-1)^n \sin^{-1}a$$

$$\cos\theta = a, \quad \theta = 2n\pi \pm \cos^{-1}a$$

$$\tan\theta = a, \quad \theta = n\pi + \tan^{-1}a$$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $(2at, at^2)$,

$$\text{tangent: } y = tx - at^2$$

$$\text{normal: } x + ty = at^3 + 2at$$

At (x_1, y_1) ,

$$\text{tangent: } xx_1 = 2a(y + y_1)$$

$$\text{normal: } y - y_1 = -\frac{2a}{x_1}(x - x_1)$$

Chord of contact from (x_0, y_0) : $xx_0 = 2a(y + y_0)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right)$$

Simple harmonic motion

$$x = b + a \cos(nt + \alpha)$$

$$\ddot{x} = -n^2(x - b)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$