Question 1

Question	Scheme	Marks	AOs
3(i)	$(5,-2)$ **or e.g.** $x=5, y=-2$ o.e.	B1	1.1b
		(1)	
(ii)	$(1.5,-2)$ **or e.g.** $x=1.5, y=-2$ o.e.	B1	1.1b
		(1)	
(iii)	$(-3, \ldots)$ **or** $(\ldots, -1)$ **or** $x=-3$ **or** $y=-1$ o.e.	B1	1.1b
	$(-3,-1)$ **or** $x=-3$ **and** $y=-1$ o.e.	B1	1.1b
		(2)	
			(4 marks)

Notes

<u>General guidelines for all parts:</u>

Remember to check answers written against the questions.
If there is any contradiction, mark the answers given in the body of the script.
If there is no labelling, mark the responses in the order given.
The coordinates need to be values not just a calculation e.g. **not** $-2\times3+5$ for -1
Points can be written as a coordinate pair or separately as $x=\ldots, y=\ldots$
Do **not** allow coordinates written the wrong way round but isw if necessary
e.g. $x=5, y=-2 \to (-2, 5)$ scores B1 and isw

Condone missing brackets (one or both) e.g. 5, −2 **or** (5, −2 **or** 5, −2) for (5, −2)
Condone a missing comma e.g. $(5\ \ -2)$ for $(5, -2)$
Condone use of a semi-colon e.g. $(5\ ;-2)$ for $(5, -2)$
Condone vector notation e.g. $\begin{pmatrix}5\\-2\end{pmatrix}$ for $(5, -2)$ and condone $\left(\dfrac{5}{-2}\right)$

(i)

B1: $(5,-2)$ o.e. see above

(ii)

B1: $(1.5,-2)$ o.e. see above

(iii)

B1: One correct coordinate. See above.

B1: Both coordinates correct. See above.
Note that B0B1 is not a possible mark profile.
Note that in part (iii), some candidates show their thinking by transforming the point piecewise e.g. $(3, -2)\to(-3, -2)\to(-3, -6)\to(-3, -1)$
In such cases, mark their **final** pair of coordinates.

Question 2

Question	Scheme	Marks	AOs
5	One of $\theta\tan 2\theta = \theta\times 2\theta$ or $1-\cos 3\theta = 1-\left(1-\frac{(3\theta)^2}{2}\right)$ or equivalents.	B1	1.1a
	$\dfrac{\theta\tan 2\theta}{1-\cos 3\theta} = \dfrac{\theta\times 2\theta}{1-\left(1-\frac{(3\theta)^2}{2}\right)}$	M1	2.1
	$=\dfrac{4}{9}$ or exact equivalent.	A1	1.1b
		(3)	
		(3 marks)	

Notes

B1: Award this mark for $\theta\tan 2\theta = \theta\times 2\theta$ or $1-\cos 3\theta = 1-\left(1-\dfrac{(3\theta)^2}{2}\right)$ or equivalents.

May be seen when working on numerator or denominator separately or within the fraction.

This is a B mark so if awarding for $\cos 3\theta$ do not condone missing brackets e.g. $1-\dfrac{3\theta^2}{2}$ unless they are recovered or are implied by subsequent work.

M1: Attempts to use both correct small angle approximations in the given expression.

For this mark they must have attempted to use $\tan 2\theta = 2\theta$ and $\cos 3\theta = 1-\dfrac{(3\theta)^2}{2}$ in the given expression but condone poor bracketing e.g. $\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)}$ or e.g. $\dfrac{\theta\times 2\theta}{1-1-\frac{3\theta^2}{2}}$

Do not allow e.g. $\dfrac{\theta\times 2\theta}{1-\frac{(3\theta)^2}{2}}$ as this suggests they are approximating $\dfrac{\theta\tan 2\theta}{\cos 3\theta}$

A1: Correct value. Do not allow rounded decimals e.g. 0.444 but allow if recurring decimals are clearly indicated e.g. $0.\dot{4}$ Do not allow e.g. $\dfrac{2}{4.5}$. Ignore any units if given.

Isw once a correct answer is seen.

<u>Examples:</u>

$\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)}=\dfrac{4}{3}\left(\text{or } -\dfrac{4}{3}\right)$	scores B1M1A0 (Missing brackets not recovered)
$\dfrac{\theta\times 2\theta}{1-\frac{(3\theta)^2}{2}}$	scores B1M0A0 (Missing "1 –" in the denominator so M0)
$\dfrac{\theta\times 2\theta}{1+\left(1-\frac{(3\theta)^2}{2}\right)}$	scores B1M0A0 (Has "1 +" in the denominator so M0)
$\dfrac{\theta\times \theta}{1-\left(1-\frac{3\theta^2}{2}\right)}=...$	scores B0M0A0 (The B mark could be recovered but M0 because of the incorrect numerator)
$\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)}=\dfrac{2\theta^2}{\frac{9\theta^2}{2}}=\dfrac{2}{9}$	scores B1M1A0 (Missing brackets recovered)
$\dfrac{\theta\times 2\theta}{1-\left(1-\left(\frac{3\theta}{2}\right)^2\right)}$	Scores B1M0A0 (The denominator suggests an incorrect expansion – unless it was recovered.)
$\dfrac{\theta\times 2\theta}{\frac{9\theta^2}{2}}=\dfrac{2}{18}$	B1M1A0 (The B1 is awarded for the numerator but can be implied by the denominator. The M1 is implied)
$\dfrac{\theta\times 2\theta}{1-\left(1-\frac{3\theta^2}{2}\right)}=\dfrac{4}{9}$	B1M1A1 (The correct value implies correct recovery of missing brackets.)

Note that other approaches are possible using identities.
In such cases we will allow <u>correct</u> work leading to an expression that if terms in θ^3 and higher can be ignored will lead to $\frac{4}{9}$
But to score the M mark they must be using correct identities and correct approximations but condone bracketing errors as in the main scheme.

Examples:

$$\frac{\theta\tan 2\theta}{1-\cos 3\theta}=\frac{\theta\times\dfrac{\sin 2\theta}{\cos 2\theta}}{1-\cos 3\theta}=\frac{\theta\times\dfrac{2\theta}{1-\dfrac{(2\theta)^2}{2}}}{1-\left(1-\dfrac{(3\theta)^2}{2}\right)}=\frac{2\theta^2}{1-2\theta^2}\times\frac{2}{9\theta^2}=\frac{4\theta^2}{9\theta^2-18\theta^4}$$

$$=\frac{4\theta^2}{9\theta^2}=\frac{4}{9}$$

Scores B1M1A1

Similarly: $\dfrac{\theta\tan 2\theta}{1-\cos 3\theta}=\dfrac{\theta\times\sin 2\theta}{\cos 2\theta(1-\cos 3\theta)}=\dfrac{\theta\times 2\theta}{\left(1-\dfrac{(2\theta)^2}{2}\right)\left(1-\left(1-\dfrac{(3\theta)^2}{2}\right)\right)}=\dfrac{2\theta^2}{1-2\theta^2}\times\dfrac{2}{9\theta^2}$ etc.

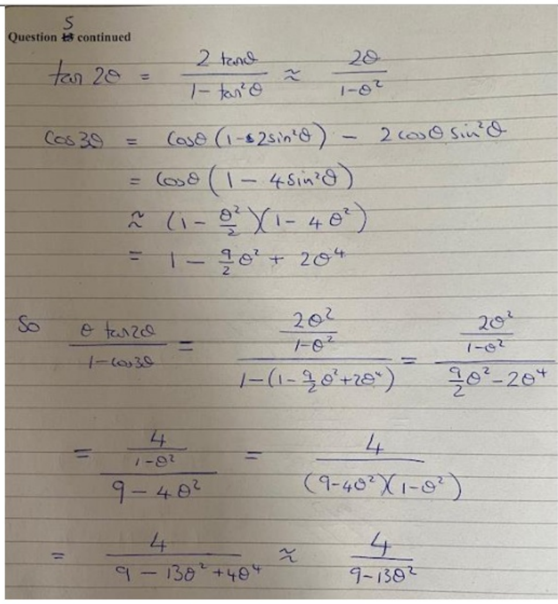
$$\frac{4\theta^2}{9\theta^2-18\theta^4}=\frac{4\theta^2}{9\theta^2}=\frac{4}{9}$$

Scores B1M1A1

$$\frac{\theta\tan 2\theta}{1-\cos 3\theta}=\frac{\theta\times\sin 2\theta}{\cos 2\theta(1-\cos 3\theta)}=\frac{\theta\times 2\theta}{\left(1-\dfrac{(2\theta)^2}{2}\right)\left(1-\left(1-\dfrac{(3\theta)^2}{2}\right)\right)}=\frac{2\theta^2}{1-2\theta^2}\times\frac{2}{9\theta^2}$$

$$=\frac{4\theta^2}{9\theta^2-18\theta^4}=\frac{4}{9-18\theta^2}=\frac{4}{9}$$

Scores B1M1A0
(They cannot just assume the term in θ^2 is 0 unless they provide a convincing limiting argument e.g. $\lim_{\theta\to 0}\frac{4}{9-18\theta^2}=\frac{4}{9}$ or equivalent)


Question 5 continued

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \approx \dfrac{2\theta}{1-\theta^2}$

$\cos 3\theta = \cos\theta(1-2\sin^2\theta) - 2\cos\theta\sin^2\theta$

$= \cos\theta(1-4\sin^2\theta)$

$\approx (1-\frac{\theta^2}{2})(1-4\theta^2)$

$= 1-\frac{9}{2}\theta^2+2\theta^4$

So $\dfrac{\theta\tan 2\theta}{1-\cos 3\theta} = \dfrac{\frac{2\theta^2}{1-\theta^2}}{1-(1-\frac{9}{2}\theta^2+2\theta^4)} = \dfrac{\frac{2\theta^2}{1-\theta^2}}{\frac{9}{2}\theta^2-2\theta^4}$

$= \dfrac{\frac{4}{1-\theta^2}}{9-4\theta^2} = \dfrac{4}{(9-4\theta^2)(1-\theta^2)}$

$= \dfrac{4}{9-13\theta^2+4\theta^4} \approx \dfrac{4}{9-13\theta^2}$

Scores B1M1A0

$$\frac{\theta\tan 2\theta}{1-\cos 3\theta}=\frac{\theta\times\dfrac{2\tan\theta}{1-\tan^2\theta}}{1-\left(4\cos^3\theta-3\cos\theta\right)}=\frac{\theta\times\dfrac{2\theta}{1-\theta^2}}{1-\left(4\left(1-\dfrac{\theta^2}{2}\right)^3-3\left(1-\dfrac{\theta^2}{2}\right)\right)}$$

Scores B1M1A0

Note that attempts to use expansions in higher powers of θ should be sent to review.

Question 3

Question	Scheme	Marks	AOs
6(a)(i)	$(\mathrm{f}'(x)=)8x\mathrm{e}^{4x^2-1}$ or e.g. $\dfrac{8x\mathrm{e}^{4x^2}}{\mathrm{e}}$ oe	B1	1.1b
(ii)	$(\mathrm{g}'(x)=)\dfrac{8}{x}$ or e.g. $8x^{-1}$ oe	B1	1.2
		(2)	
(b)	$8x\mathrm{e}^{4x^2-1}=\dfrac{8}{x}\Rightarrow \mathrm{e}^{4x^2-1}=\dfrac{1}{x^2}\Rightarrow 4x^2-1=\ln\dfrac{1}{x^2}$	M1	1.1b
	$4x^2-1=\ln\dfrac{1}{x^2}\Rightarrow 4x^2-1=-2\ln x$ $\Rightarrow 4x^2+2\ln x-1=0$ *	A1*	2.1
		(2)	
(c)(i)	$x_1=0.6\Rightarrow x_2=\sqrt{\dfrac{1-2\ln 0.6}{4}}$	M1	1.1b
	$(x_2=)0.7109$	A1	1.1b
(ii)	$(\alpha=)0.6706$	B1 (A1 In ePEN)	1.1b
		(3)	
			(7 marks)

(a)(i)

B1: Correct derivative in any form. "$\mathrm{f}'(x)=$" is not required. Apply isw if necessary.

(ii)

B1: Correct derivative in any form. "$\mathrm{g}'(x)=$" is not required. Apply isw if necessary.

(b)

M1: Eliminates e by setting their $\mathrm{f}'(x)=$ their $\mathrm{g}'(x)$ where $\mathrm{f}'(x)=Ax\mathrm{e}^{4x^2-1}$ oe and $\mathrm{g}'(x)=\dfrac{B}{x}$ oe with $A\times B>0$ and proceeds via $\mathrm{e}^{4x^2-1}=\dfrac{...}{x^2}$ or equivalent work (see below) to obtain $4x^2-1=\ln\dfrac{...}{x^2}$ oe e.g. $\ln x+4x^2-1=\ln\dfrac{1}{x}$

Allow if they use α for x.

Note that there are various alternatives for this mark but the derivatives must be of the form defined above and the processing must be correct with coefficient/sign slips only.

Examples of equivalent work:

$$8x\mathrm{e}^{4x^2-1}=\frac{8}{x}\Rightarrow x^2\mathrm{e}^{4x^2-1}=1\Rightarrow \ln x^2+\ln\mathrm{e}^{4x^2-1}=0\Rightarrow \ln\mathrm{e}^{4x^2-1}=-\ln x^2\Rightarrow 4x^2-1=-2\ln x$$

$$\frac{8x\mathrm{e}^{4x^2}}{\mathrm{e}}=\frac{8}{x}\Rightarrow\frac{1}{\mathrm{e}}\mathrm{e}^{4x^2}=\frac{1}{x^2}\Rightarrow \mathrm{e}^{4x^2}=\frac{\mathrm{e}}{x^2}\Rightarrow \ln\mathrm{e}^{4x^2}=\ln\frac{\mathrm{e}}{x^2}\Rightarrow 4x^2=\ln\frac{\mathrm{e}}{x^2}=1-2\ln x$$

A1*: Obtains the printed answer with sufficient working and no errors.
Sufficient work would require the "e" eliminated before the given answer.
Must follow correct derivatives in part (a).
Condone $4x^2+2\ln|x|-1=0$ and condone $4\alpha^2+2\ln\alpha-1=0$ or $4\alpha^2+2\ln|\alpha|-1=0$

Note that if both derivatives in (a) **are correct** we will allow fully correct work using the equation in (b) to work backwards to verify that $p\mathrm{f}'(x)=q\mathrm{g}'(x)$ for M1 then obtains $\mathrm{f}'(x)=\mathrm{g}'(x)$ with a minimal conclusion for A1

If either derivative in (a) is incorrect or missing, candidates who work backwards score no marks in (b).

(c)(i)/(ii)

M1: Attempts to use the iterative formula with $x_1=0.6$

Award this mark for e.g. $(x_2 =)\sqrt{\dfrac{1-2\ln 0.6}{4}}$ or may be implied by awrt 0.71 provided no incorrect working is seen.
Candidates sometimes find x_3 (or possibly subsequent terms) rather than x_2 in which case the M1 can be implied. (See table below for first few iterations)

A1: (x_2 =) awrt 0.7109
Sight of (x_2 =) awrt 0.7109 scores M1A1

B1(A1 on ePEN): (α =) 0.6706 (4dp)
Must be <u>this value</u> and **not** awrt 0.6706

For reference:

x_1	0.6
x_2	0.7109239143
x_3	0.6485329086
x_4	0.6830236199
x_5	0.6637868021
x_6	0.6744606223
. . .	. . .
α	0.6706416243

Question 4

Question	Scheme	Marks	AOs
8(a) Way 1	$\left(\dfrac{1}{\operatorname{cosec}\theta-1}+\dfrac{1}{\operatorname{cosec}\theta+1}\equiv\right)\dfrac{\operatorname{cosec}\theta+1+\operatorname{cosec}\theta-1}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}$	B1	1.1b
	$\equiv\dfrac{2\operatorname{cosec}\theta}{\operatorname{cosec}^2\theta-1}\equiv\dfrac{2\operatorname{cosec}\theta}{\cot^2\theta}$ or e.g. $\equiv\dfrac{2\sin\theta}{1-\sin^2\theta}=\dfrac{2\sin\theta}{\cos^2\theta}$	M1	1.1b
	$\dfrac{2\operatorname{cosec}\theta}{\cot^2\theta}\equiv\dfrac{2}{\sin\theta}\times\dfrac{\sin^2\theta}{\cos^2\theta}\equiv 2\tan\theta\sec\theta*$ or $\dfrac{2\sin\theta}{\cos^2\theta}\equiv 2\tan\theta\sec\theta*$	A1*	2.1
		(3)	
(a) Way 2	**"Meets in the middle"**		
	$\left(\text{LHS}=\dfrac{1}{\operatorname{cosec}\theta-1}+\dfrac{1}{\operatorname{cosec}\theta+1}\equiv\right)\dfrac{\operatorname{cosec}\theta+1+\operatorname{cosec}\theta-1}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}$	B1	1.1b
	$\equiv\dfrac{2\operatorname{cosec}\theta}{\operatorname{cosec}^2\theta-1}\equiv\dfrac{2\operatorname{cosec}\theta}{\cot^2\theta}$	M1	1.1b
	$\text{RHS}=2\tan\theta\sec\theta\equiv\dfrac{2\sin\theta}{\cos^2\theta}\equiv\dfrac{2\sin^2\theta}{\sin\theta\cos^2\theta}$ $\equiv\dfrac{2}{\sin\theta}\times\dfrac{\sin^2\theta}{\cos^2\theta}\equiv\dfrac{2\operatorname{cosec}\theta}{\cot^2\theta}=\text{LHS}$ or e.g. QED or e.g. Proven	A1*	2.1
		(3)	

Part (a) Notes

(a) **Condone a complete proof entirely in x (or another variable) instead of θ**
Condone "=" for "≡"

Note that we are marking this as **B1M1A1** not **M1M1A1**

B1: Adds the fractions to obtain a **correct** single fraction (not fractions over fractions) in any form. Condone missing brackets when they combine their fractions as long as they are recovered to give a correct fraction.

This can be done in a variety of ways but when combined, the fraction must be **correct** e.g.

$$\frac{\operatorname{cosec}\theta+1+\operatorname{cosec}\theta-1}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}\text{ or }\frac{2\operatorname{cosec}\theta}{(\operatorname{cosec}\theta-1)(\operatorname{cosec}\theta+1)}\text{ or }\frac{2\operatorname{cosec}\theta}{\operatorname{cosec}^2\theta-1}$$

$$\text{or e.g. }\left(\frac{1}{\frac{1}{\sin\theta}-1}+\frac{1}{\frac{1}{\sin\theta}+1}=\frac{\sin\theta}{1-\sin\theta}+\frac{\sin\theta}{1+\sin\theta}\right)=\frac{2\sin\theta}{1-\sin^2\theta}\text{ etc.}$$

M1: **Uses** a **correct** Pythagorean identity anywhere in their attempt e.g.
$\operatorname{cosec}^2\theta-1=\cot^2\theta$, $\sin^2\theta+\cos^2\theta=1$ etc. or equivalent

A1*: Correct work with all necessary steps shown leading to the given answer. See scheme for the necessary steps. They need to proceed via sine and cosine to the given answer. There should be **no notational or bracketing errors and no mixed or missing variables**. E.g. we would consider $\cos^2\theta$ written as $\cos\theta^2$ a notational error.
Condone reaching $2\sec\theta\tan\theta*$

Way 2 (Meet in the middle)

B1: See Way 1
M1: See Way 1
A1*: Correct work on the RHS with all necessary steps shown leading to showing the equivalence with the LHS. See scheme for the necessary steps. There should be no notational or bracketing errors and no mixed or missing variables.
For this approach there must be a (minimal) conclusion e.g. "= LHS", "QED", "Hence proven" etc.

It is possible to start with the rhs e.g.:

$$2\tan\theta\sec\theta = 2\frac{\sin\theta}{\cos\theta}\times\frac{1}{\cos\theta}$$
$$= \frac{2\sin\theta}{\cos^2\theta} = \frac{2\text{cosec}\,\theta}{\cot^2\theta}$$
$$= \frac{2\text{cosec}\,\theta}{\text{cosec}^2\theta - 1} = \frac{2\text{cosec}\,\theta}{(\text{cosec}\,\theta-1)(\text{cosec}\,\theta+1)}$$
$$= \frac{\text{cosec}\,\theta + 1 + \text{cosec}\,\theta - 1}{(\text{cosec}\,\theta-1)(\text{cosec}\,\theta+1)}$$
$$= \frac{1}{\text{cosec}\,\theta - 1} + \frac{1}{\text{cosec}\,\theta+1}$$

B1: Correctly reaches $2\tan\theta\sec\theta = \frac{2\text{cosec}\,\theta}{(\text{cosec}\,\theta-1)(\text{cosec}\,\theta+1)}$

M1: **Uses** a **correct** Pythagorean identity anywhere in their attempt e.g. $\text{cosec}^2\theta - 1 = \cot^2\theta$, $\sin^2\theta + \cos^2\theta = 1$ etc. or equivalent

A1*: Correct work with all necessary steps shown leading to the lhs. See scheme for the necessary steps. There should be no notational or bracketing errors and no mixed or missing variables.

8(b)	$2\tan 2x\sec 2x = \cot 2x\sec 2x$	B1	2.2a
	$2\tan 2x\sec 2x - \cot 2x\sec 2x = 0$ $\Rightarrow \sec 2x(2\tan 2x - \cot 2x) = 0$ $2\tan 2x - \cot 2x = 0 \Rightarrow 2\tan 2x = \cot 2x \Rightarrow \tan^2 2x = \frac{1}{2}$ **or** $2\tan 2x - \cot 2x = 0 \Rightarrow 2\frac{\sin 2x}{\cos 2x} = \frac{\cos 2x}{\sin 2x} \Rightarrow 2\sin^2 2x = \cos^2 2x$ $\Rightarrow 2\sin^2 2x = 1 - \sin^2 2x \Rightarrow \sin^2 2x = \frac{1}{3}$ **or** $2(1-\cos^2 2x) = \cos^2 2x \Rightarrow \cos^2 2x = \frac{2}{3}$ **or** $2\tan 2x = \cot 2x \Rightarrow \frac{4\tan x}{1-\tan^2 x} = \frac{1-\tan^2 x}{2\tan x}$ $\Rightarrow \tan^4 x - 10\tan^2 x + 1 = 0$ **or** $2\tan 2x - \cot 2x = 0 \Rightarrow 2\frac{\sin 2x}{\cos 2x} = \frac{\cos 2x}{\sin 2x} \Rightarrow 2\sin^2 2x = \cos^2 2x$ $2\sin^2 2x = \cos^2 2x \Rightarrow 1 - \cos 4x = \frac{1}{2}(\cos 4x + 1) \Rightarrow \cos 4x = \frac{1}{3}$	M1	2.1
	$2x = \tan^{-1}\frac{1}{\sqrt{2}} = K \Rightarrow x = \frac{K}{2}$ **or** $2x = \sin^{-1}\frac{1}{\sqrt{3}} = K \Rightarrow x = \frac{K}{2}$ **or** $2x = \cos^{-1}\sqrt{\frac{2}{3}} = K \Rightarrow x = \frac{K}{2}$ **or** $\tan^2 x = 5 \pm 2\sqrt{6} \Rightarrow x = \tan^{-1}\left(\sqrt{5\pm2\sqrt{6}}\right)$ **or** $\cos 4x = \frac{1}{3} \Rightarrow 4x = \cos^{-1}\left(\frac{1}{3}\right) = K \Rightarrow x = \frac{1}{4}K$	M1	1.1b
	$x = 17.6°, 72.4°$	A1	1.1b
		(4)	

(7 marks)

(b) Notes

(b) Note that attempts solve an equation of the form:

$2\tan x\sec x = \cot 2x\sec 2x$ or e.g. $2\tan\theta\sec\theta = \cot 2\theta\sec 2\theta$ or e.g. $2\tan\theta\sec\theta = \cot 2x\sec 2x$

Will generally score no marks in part (b)

Condone the use of θ instead of *x* here.

B1: Deduces the correct equation using the result from part (a)

M1: Factors out or cancels the $\sec 2x$ to obtain $...\tan 2x \pm ...\cot 2x = 0$ oe e.g. $...\tan 2x = \pm ...\cot 2x$ leading to an equation of the form:

$\tan^2 2x = \alpha$ oe e.g. $\cot^2 2x = \dfrac{1}{\alpha}$ where $\alpha > 0$

Or uses $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$ and $\cot 2x = \dfrac{\cos 2x}{\sin 2x}$ and then $\sin^2 2x = \pm 1 \pm \cos^2 2x$ or $\cos^2 2x = \pm 1 \pm \sin^2 2x$ to obtain an equation of the form $\sin^2 2x = \beta$ or $\cos^2 2x = \gamma$ oe

e.g. $\operatorname{cosec}^2 2x = \dfrac{1}{\beta}$ or $\sec^2 2x = \dfrac{1}{\gamma}$ where $0 < \beta < 1$ or $0 < \gamma < 1$

Or uses $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$ and $\cot 2x = \dfrac{1-\tan^2 x}{2\tan x}$ to obtain a 3TQ in $\tan^2 x$ (or possibly in $\sec^2 x$)

Or uses $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$ and $\cot 2x = \dfrac{\cos 2x}{\sin 2x}$ and then $2\sin^2 2x = \pm 1 \pm \cos 4x$ and $2\cos^2 2x = \pm 1 \pm \cos 4x$ to obtain an equation of the form $\cos 4x = k,\ \ 0 < k < 1$

M1: Correct order of operations from $\tan^2 2x = \alpha$ or $\sin^2 2x = \beta$ or $\cos^2 2x = \gamma$ or $\cos 4x = k$ or equivalents e.g. $\operatorname{cosec}^2 2x = \dfrac{1}{\beta}$ where $\alpha > 0$ or $0 < \beta < 1$ or $0 < \gamma < 1$ or $0 < k < 1$

leading to at least one value for *x* e.g. square roots, finds inverse tan/sin/cos/cosec and divides by 2 or inverse cos and divides by 4

or from $\tan^2 x = k,\ k > 0$ which follows their equation (may need to check) and then finds $x = \tan^{-1}\sqrt{k}$

You may need to check their value(s) (in degrees or radians) to see if the correct order of operations has been used. May be implied by e.g. $17.6°$ or $17.7°$ provided no incorrect work is seen.

A1: Correct values. Allow awrt $17.6°$ and awrt $72.4°$. The degrees symbol is not required. Ignore any values outside the range (correct or incorrect) but if there are extra angles in range score A0. Answers in radians score A0.

Note that some candidates may convert to $\sin x$ or $\cos x$ and then solve:

E.g.

$$\tan^2 2x = \frac{1}{2} \Rightarrow \frac{\sin^2 2x}{\cos^2 2x} = \frac{1}{2} \Rightarrow \frac{4\sin^2 x\cos^2 x}{\left(2\cos^2 x - 1\right)^2} \to 12\cos^4 x - 12\cos^2 x + 1 = 0$$

$$\text{or } \frac{4\sin^2 x\cos^2 x}{\left(1 - 2\sin^2 x\right)^2} \to 12\sin^4 x - 12\sin^2 x + 1 = 0$$

$$\Rightarrow \cos^2 x / \sin^2 x = \frac{3 \pm \sqrt{6}}{6} \Rightarrow \cos x / \sin x = \pm\sqrt{\frac{3 \pm \sqrt{6}}{6}} \Rightarrow x = 17.6°,\ 72.4°$$

These can be marked in a similar way.

<u>Alternative not using part (a):</u>

$$\frac{1}{\quad} + \frac{1}{\quad} = \cot 2x\sec 2x$$

$$\text{cosec}2x-1 \quad \text{cosec}2x+1$$

$$\Rightarrow \frac{2\text{cosec}2x}{\text{cosec}^2 2x-1}=\cot 2x \sec 2x$$

$$\Rightarrow \frac{2\text{cosec}2x}{\cot^2 2x}=\frac{\cos 2x}{\sin 2x}\times\frac{1}{\cos 2x}=\text{cosec}2x$$

$$\Rightarrow 2\tan^2 2x=1 \Rightarrow \tan^2 2x=\frac{1}{2}$$

Score as:

M1: For correct work leading to one of the forms in the main scheme e.g.

$$\tan^2 2x=\alpha \text{ oe e.g. } \cot^2 2x=\frac{1}{\alpha} \text{ where } \alpha>0$$

or

$$\sin^2 2x=\beta \text{ or } \cos^2 2x=\gamma \text{ oe e.g. } \text{cosec}^2 2x=\frac{1}{\beta} \text{ or } \sec^2 2x=\frac{1}{\gamma}$$

where $0<\beta<1$ or $0<\gamma<1$

B1: Any correct equation e.g. $\tan^2 2x=\frac{1}{2}$, $\cot^2 2x=2$, $\sin^2 2x=\frac{1}{3}$ etc.

Then **M1A1** as main scheme

Question 5

Question	Scheme	Marks	AOs
12(a)	$\dfrac{1}{V(25-V)} = \dfrac{P}{V} + \dfrac{Q}{25-V}$ e.g. $1 = P(25-V) + QV$ $V = 0$ or $V = 25$ leading to $P = \ldots$ or $Q = \ldots$	M1	1.1b
	$\dfrac{1}{V(25-V)} = \dfrac{1}{25V} + \dfrac{1}{25(25-V)}$	A1	1.1b
		(2)	
Notes			
(a) **M1:** Sets $\dfrac{1}{V(25-V)} = \dfrac{P}{V} + \dfrac{Q}{25-V}$ and uses a correct method to identify the value of at least one constant. Do **not** condone incorrect work e.g. $\dfrac{1}{V(25-V)} = \dfrac{P}{V} + \dfrac{Q}{25-V} \Rightarrow 1 = PV + Q(25-V)$ etc. this scores M0 **A1:** Correct partial fractions in any form e.g. $\dfrac{1}{25V} + \dfrac{1}{25(25-V)}, \dfrac{1}{25V} + \dfrac{1}{625-25V}, \dfrac{1/25}{V} + \dfrac{1/25}{(25-V)}, \dfrac{1}{25V} - \dfrac{1}{25(V-25)}, \dfrac{1}{25}\left(\dfrac{1}{V} + \dfrac{1}{25-V}\right)$ etc. Note that this mark is not just for the correct constants, it is for the correctly written fractions either seen in part (a) **or used in part (b).** Allow 0.04 for $\dfrac{1}{25}$. Correct partial fractions only scores both marks. If the correct fractions are obtained following incorrect work score M0A0 but allow full recovery in the rest of the question.			
(b) **Way 1**	$\displaystyle\int \frac{1}{V(25-V)}\,dV = \int \frac{1}{25V} + \frac{1}{25(25-V)}\,dV = \frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)$	M1	3.1a
	$\dfrac{1}{25}\ln V - \dfrac{1}{25}\ln(25-V) = \dfrac{1}{10}t\,(+c)$	A1ft	1.1b
	$t = 0, V = 20 \Rightarrow \dfrac{1}{25}\ln 20 - \dfrac{1}{25}\ln(25-20) = c\left(\Rightarrow c = \dfrac{1}{25}\ln 4\right)$	M1	3.4
	$V = 24 \Rightarrow t = \dfrac{2}{5}\ln 24 - \dfrac{2}{5}\ln(25-24) - \dfrac{2}{5}\ln 4$	dM1	3.1b
	$= 43$ (or exact $24\ln 6$)	A1	3.2a
		(5)	
	Alternative for the final 3 marks:		
	$\left[\dfrac{1}{25}\ln V - \dfrac{1}{25}\ln(25-V)\right]_{20}^{24} = \left[\dfrac{1}{10}t\right]_0^T \Rightarrow \dfrac{1}{25}\ln 24 - \dfrac{1}{25}\ln 4 = \dfrac{1}{10}T$	M1	3.4
	$T = \dfrac{10}{25}\ln 24 - \dfrac{10}{25}\ln 4 = \ldots$	dM1	3.1b
	$= 43$ (or exact $24\ln 6$)	A1	3.2a
(c)	$\dfrac{1}{25}\ln V - \dfrac{1}{25}\ln(25-V) = \dfrac{1}{10}t + \dfrac{1}{25}\ln 4$ $\ln V - \ln(25-V) = 2.5t + \ln 4$ $\ln\dfrac{V}{4(25-V)} = 2.5t \Rightarrow \dfrac{V}{4(25-V)} = e^{2.5t}$	M1	2.1
	$\Rightarrow V = 4e^{2.5t}(25-V) \Rightarrow V + 4Ve^{2.5t} = 100e^{2.5t} \Rightarrow V = \ldots$	M1	2.1
	$\Rightarrow V = \dfrac{100e^{2.5t}}{1+4e^{2.5t}} = \dfrac{100}{e^{-2.5t}+4}$	A1	1.1b
		(3)	

(d)	25 (microlitres)	B1	2.2a
	Since e.g. As $t \to \infty$, $e^{"-2.5"t} \to 0$	B1	2.4
		(2)	
			(12 marks)

Notes

(b) **Mark (b) and (c) together**

M1: Realises that $\int \frac{...}{V(25-V)}(\mathrm{d}V)$ is required and reaches the form $p\ln\alpha V \pm q\ln\beta(25-V)$ (or e.g. $p\ln\alpha V \pm q\ln\beta(V-25)$) or equivalent for this integration e.g. $p\ln 25V - q\ln(625-25V)$ with p and q non-zero.

But note that $\int \frac{...}{V(25-V)}\mathrm{d}V = \ln V(25-V)$ does not score this mark unless we see an attempt to integrate the partial fractions first.
Condone missing brackets e.g. around the $V - 25$ for this mark
Note that the rhs may be incorrect or missing for this mark.

A1ft: Fully correct <u>**equation**</u> following through their P and Q. The "+ c" is **not** required here.
You may need to check carefully when awarding this mark as there will be various alternative correct (or correct ft) forms e.g. these are correct (for correct PF's):

$\frac{1}{25}\ln 25V - \frac{1}{25}\ln(625-25V) = \frac{1}{10}t(+c)$, $\ln V - \ln(25-V) = 2.5t(+c)$,
$\frac{2}{5}\ln 25V - \frac{2}{5}\ln(25-V) = t(+c)$, $\frac{2}{5}\ln 5V - \frac{2}{5}\ln(125-5V) = t(+c)$

In general look for an equation of the form $P\ln\alpha V - Q\ln\beta(25-V) = \frac{1}{10}t(+c)$ or a multiple of this equation. Do **not** condone missing brackets unless they are implied by later work e.g. $\ln 25 - V$ for $\ln(25-V)$

Allow ( ) or | | around the arguments of the ln's and condone "log" for ln.

M1: States or uses $t = 0$ and $V = 20$ consistently leading to a constant of integration which may be simplified or unsimplified. May be implied by their constant so may need to be checked.
This mark is not formally dependent but depends on having made some attempt to integrate both sides, however poor.

dM1: States or uses $V = 24$ and proceeds to find a value for t (even if $t < 0$). You do not need to check the processing provided they reach a value for t. **Depends on the previous method mark** and depends on an attempt to integrate both sides however poor.
May be implied by their value for t so may need to be checked.

A1: Correct value of 43 or awrt 43.0 or exact value of 24ln6.
Units are not required but if any are given it must be minutes or condone "m".
Note that in hours the time is 0.7167037877… and scores A0

Alternative for final 3 marks:

M1: $\left[\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V)\right]_{20}^{24} = \left[\frac{1}{10}t\right]_0^T \Rightarrow \frac{1}{25}\ln 24 - \frac{1}{25}\ln 4 = \frac{1}{10}T$
Applies the limits 20 and 24 to lhs and 0 to "T" or e.g. "t" on rhs
This mark is not formally dependent but depends on having made some attempt to integrate both sides, however poor.

dM1: $T = \frac{10}{25}\ln 24 - \frac{10}{25}\ln 4 = ...$ Solves to find "T". You do not need to check the processing provided they reach a value for t/T. **Depends on the previous method mark** and depends on having made some attempt to integrate both sides however poor.

A1: Correct value of 43 or awrt 43.0 or exact value of 24ln6.
Units are not required but if any are given it must be minutes or condone "m".
Note that in hours the time is 0.7167037877… and scores A0

<u>**Note**</u>

$$\int \frac{1}{25V} + \frac{1}{25(25-V)}\mathrm{d}V = \int \frac{1}{25V} - \frac{1}{25V-625}\mathrm{d}V = \frac{1}{25}\ln V - \frac{1}{25}\ln(25V-625) = \frac{1}{10}t(+c)$$

is also correct integration and scores M1A1

The subsequent marks are also available as described above and could lead to the correct answer if the limits and constant of integration are dealt with correctly.
Use review for any examples like these if you are unsure but generally apply the MS as above.

(c) The marks in (c) depend on having integrated their partial fractions to obtain an equation of the form $\pm ...\ln ...V \pm ...\ln ...(25-V) = \pm kt \pm c, \quad k, c \neq 0$ and … are non-zero constants or equivalent if they have already attempted to eliminate the ln's in (b)

e.g. $\frac{...V}{...(25-V)} = \text{e}^{\pm kt \pm c}$ oe

M1: Uses fully correct log work, having obtained a constant of integration, to eliminate all the ln's including from e.g. $\text{e}^{\ln 4}$. We condone sign or coefficient slips only.

M1: Proceeds from an **equation of the form** $\frac{...V}{...(25-V)} = ...\text{e}^{...t}$ oe using correct algebra to

$V = …$ e.g. $\frac{...V}{...(25-V)} = ...\text{e}^{...t} \Rightarrow ...V = ...(25-V)...\text{e}^{...t} \Rightarrow (...\pm...)V = ...\text{e}^{...t} \Rightarrow V = ...$

Condone sign/coefficient slips only.

A1: Correct expression not just values for the constants.

(d)

B1: Correct value of 25 seen

Allow e.g. < 25 or $\leqslant 25$

Condone "> 25" but the following mark is then not available

B1: Depends on a correct final equation in any form in (c) e.g. $V = \frac{100\text{e}^{2.5t}}{1+4\text{e}^{2.5t}}$ oe **and one of**:

- Considers the behaviour as $t \to \infty$ e.g. states that as $t \to \infty$, $\text{e}^{-2.5t} \to 0$ (condone "= 0") oe
- $V < 25$ as $\ln(25 - V)$ is not possible when $V \geqslant 25$
- Verifies the 25 using a value of t, $t \geqslant 9$

Using the differential equation:

B1: Correct value of 25 seen

Allow e.g. "< 25" or $\leqslant 25$

Condone "> 25" but the following mark is then not available

B1: E.g. when $V = 25$, $\frac{dV}{dt} = 0$ **or** $\frac{dV}{dt} < 0$ if $V > 25$

Question 6

Question	Scheme	Marks	AOs
13(a)	$\log_{10} b = 0.0054 \Rightarrow b = 10^{0.0054}$ **or** $\log_{10} a = 0.81 \Rightarrow a = 10^{0.81}$	M1	3.1a
	$b = 1.01$ **or** $a = 6.46$	A1	1.1b
	$\log_{10} b = 0.0054 \Rightarrow b = 10^{0.0054}$ **and** $\log_{10} a = 0.81 \Rightarrow a = 10^{0.81}$	M1	2.1
	$b = 1.013$ **and** $a = 6.457$	A1	1.1b
		(4)	
(b)(i)	e.g. The world population **in billions** in 2004	B1ft	3.2a
(ii)	b = 1.013 represents the scale factor of the <u>yearly increase</u> in the world population	B1ft	3.2a
		(2)	
(c)	$P = 6.457...(1.013...)^{26}$ or e.g. $\log P = 0.81 + 26 \times 0.0054 \Rightarrow P = ...$	M1	3.4
	awrt 9 billion	A1	2.2b
		(2)	
(d)	Not reliable since the data used for the model covered the years 2004 – 2007 and it would not be sensible to assume that the model still holds in 2030	B1	3.2b
		(1)	
			(9 marks)

Notes

(a) Must be using base 10 in (a). Ignore any units associated with a and b in part (a).

M1: Correct strategy to get a numerical expression or value for a **or** b e.g. $a = 10^{0.81}$ **or** $b = 10^{0.0054}$. This may be implied by a = awrt 6.46 or b = awrt 1.01 if no incorrect work is seen.

A1: Correct value for a or b. Allow 3 sf for **this mark** so allow a = awrt 6.46 or b = awrt 1.01.
May be seen embedded in their formula.

M1: Correct strategy to get a numerical expression or value for a **and** b e.g. $a = 10^{0.81}$ **and** $b = 10^{0.0054}$. This may be implied by a = awrt 6.46 and b = awrt 1.01 if no incorrect work is seen.

A1: Correct values. This requires a = awrt 6.457 and b = awrt 1.013 for this mark.
May be seen embedded in their formula.
Isw once correct answers are seen.

<u>Special case</u>: Constants the wrong way round:

$a = 1.013$ **and** $b = 6.457$ with or without working scores M1A1M1A0 unless the equation is formed correctly in which case the final A mark can be recovered.

Note that having found the value of a, it is possible to find b by substituting e.g. t = 1 as follows:

$$a = 10^{0.81} = 6.457 \quad t = 1 \Rightarrow P = ab \Rightarrow b = \frac{P}{a}$$

$$t = 1 \Rightarrow \log_{10} P = 0.0054 + 0.81 = 0.8154 \Rightarrow P = 10^{0.8154} \Rightarrow b = \frac{P}{a} = \frac{10^{0.8154}}{6.457} = 1.013$$

Note that a misread of 0.0054 as 0.054 is quite common and may score 1110 as it does not simplify the question.

(b)(i) Follow through their a.

B1ft: Correct interpretation for a but must reference "billions".
Allow equivalent alternatives e.g.

- The original/initial population **in billions**
- The population in 2004 was "6.46" **billion**

(b)(ii) Follow through their b.

B1ft: Correct interpretation for b but must reference "each year" or e.g. "yearly" oe
Allow equivalent alternatives e.g.

- The proportional increase/change in each year.
- The population will rise by "1.3%" each year. Must follow their value for b.
- The rate/factor at which the population is rising/increasing/changing per annum.
- "1.013" is the multiplier representing the year on year increase.

Do **not** accept

- The amount it is rising
- How much it is rising
- The rate the population increases
- The percentage increase each year
- The rate of increase in billions annually

If they are not labelled (b)(i) and (b)(ii) mark in the order given but accept any way round as long as clearly labelled " *a* is........... " and "*b* is ...."

(c)

M1: Substitutes $t = 25$ or 26 or 27 into their model to find a value for P
Must be using **their** a and b correctly in $P = ab^t$
May be implied by sight of "9" or 9 billion if no incorrect working is seen.

A1: Correct value including units (allow awrt 9 billion) **from a correct model but condone incorrect/premature rounding or truncating in an otherwise correct model that leads to the correct value of awrt 9 billion.**
Allow e.g. awrt 9 000 000 000 or e.g. awrt 9×10^9
Just awrt 9 without the "billions" is A0

(d)

B1: The response must refer to the fact that the answer is unreliable together with a reference to the fact that the data used for the model is a long way from 2030
Examples:

- Not good as 2030 is a long way from 2004 – 2007
- Unreliable as based on old data
- Questionable as it has been extrapolated over a long time
- Not reliable due to how far out we have extrapolated
- By the time 2030 arrives it will be unreliable

But not e.g.

- Unreliable, extrapolation
- Not good as outside the range
- Not good as the population rises 101.3% each year
- Disease may happen
- Reliable as based on old data

Question 7

Question	Scheme	Marks	AOs
3(a)	Uses one correct log law e.g. $\log_2(x+3)+\log_2(x+10)=\log_2(x+3)(x+10)$ $2=\log_2 4,\quad 2\log_2 x=\log_2 x^2$	M1	1.1b
	$(x+3)(x+10)=4x^2$ oe	dM1	2.1
	$\Rightarrow 3x^2-13x-30=0$ *	A1*	1.1b
		(3)	
(b)(i)	$(x=)\ 6,\ -\dfrac{5}{3}$	B1	1.1b
(ii)	$x\neq-\dfrac{5}{3}$ because $\log_{(2)}\left(-\dfrac{5}{3}\right)$ is not real	B1	2.4
		(2)	
		(5 marks)	

Notes

(a)

M1: Uses one correct log law. The base does not need to be seen for this mark.
This mark is independent of any other errors they make.
Examples: $\log_2(x+3)+\log_2(x+10)=\log_2(x+3)(x+10)$, $2=\log_2 4$, $2\log_2 x=\log_2 x^2$

dM1: Fully correct work leading to a correct equation not containing logs that is not the printed answer. **Depends on the first mark.** Condone a spurious base e.g. 10 or e, so long as the log work is otherwise correct (i.e., they recover the base 2).
Examples (depending on their method): $(x+3)(x+10)=4x^2$, $\dfrac{(x+3)(x+10)}{x^2}=4$, $\dfrac{x+3}{x^2}=\dfrac{4}{x+10}$ or $1+\dfrac{13}{x}+\dfrac{30}{x^2}=4$

Allow recovery from invisible brackets but **not** from incorrect work e.g.

$$\log_2(x+3)+\log_2(x+10)=2+2\log_2 x\Rightarrow\log_2(x+3)(x+10)-\log_2 x^2=\log_2 4$$

$$\Rightarrow\frac{\log_2(x+3)(x+10)}{\log_2 x^2}=\log_2 4\Rightarrow\frac{(x+3)(x+10)}{x^2}=4$$

This scores M1dM0A0

A1*: Obtains $3x^2-13x-30=0$ with no processing errors but condone a spurious base e.g. 10 or e, so long as the log work is otherwise correct (i.e., they recover the base 2) and allow recovery from invisible brackets.

Note the following alternative which can follow the main scheme:

$\log_2(x+3)+\log_2(x+10)=2+2\log_2 x=2+\log_2 x^2$ **M1**

$2^{\log_2(x+3)+\log_2(x+10)}=2^{2+\log_2 x^2}\Rightarrow 2^{\log_2(x+3)}\times 2^{\log_2(x+10)}=2^2\times 2^{\log_2 x^2}\Rightarrow(x+3)(x+10)=4x^2$ **dM1**

$\Rightarrow 3x^2-13x-30=0$ * **A1**

Special Cases:

1. $(x+3)(x+10)=4x^2$ with no working leading to the correct answer scores **M1dM1A0**

2. $\log_2(x+3)+\log_2(x+10)=2+2\log_2 x\Rightarrow 2^{\log_2(x+3)+\log_2(x+10)}=2^{2+2\log_2 x}\Rightarrow(x+3)(x+10)=4x^2$

$\Rightarrow 3x^2-13x-30=0$ *

Also scores **M1(implied)dM1A0** (lack of working)

(b)(i)

B1: Both values correct: $(x=)\ 6,\ -\dfrac{5}{3}$

(b)(ii)

B1: e.g. $(x\neq)-\dfrac{5}{3}$ and $\log_{(2)}\left(-\dfrac{5}{3}\right)$ is not real

This mark requires the identification of the **correct** negative root **and** an acceptable explanation.

For the identification of the root allow e.g. $x \neq -\frac{5}{3}$, $x = -\frac{5}{3}$, $-\frac{5}{3}$ etc. as long as it is clear they have identified the correct value. Requires the correct negative root $\left(-\frac{5}{3}\right)$ but the other may not be 6 but must be positive.

Some examples for the explanation:

- you get $\log_{(2)}\left(-\frac{5}{3}\right)$ which is not possible
- $\log -\frac{5}{3}$ is not possible, can't be found, gives a math error, is not real, is undefined
- if $\left\{k = \log_2\left(-\frac{5}{3}\right),\right\}$ $2^k = -\frac{5}{3}$ which is not possible
- you get log of a negative number
- negative numbers can't be "logged"
- log of negative does not work

Do not allow e.g.

- you can't have a negative log, logs can't be negative (unless clarified further)
- "you get a math error" in isolation
- a log cannot have a negative value
- logs cannot be negative
- $-\frac{5}{3}$ is not a valid input (unless clarified further)
- "it doesn't work in the logs"
- log graph isn't negative
- log graph does not cross negative x–axis
- x is only positive & negative answer does not work

Allow an implied correct answer if they say e.g. 6 is the root because $\log_{(2)}\left(-\frac{5}{3}\right)$ is not possible

Question 8

Question	Scheme	Marks	AOs
7(a)	$x^3 \to ...x^2$ and $3y^2 \to ...y\frac{dy}{dx}$	M1	1.1b
	$2xy \to 2y + 2x\frac{dy}{dx}$	B1	1.1b
	$3x^2 + 2x\frac{dy}{dx} + 2y + 6y\frac{dy}{dx} = ... \Rightarrow \frac{dy}{dx} = ...$	M1	2.1
	$\frac{dy}{dx} = -\frac{2y+3x^2}{2x+6y}$	A1	1.1b
		(4)	
(b)	$\frac{dy}{dx} = -\frac{2(5)+3(-2)^2}{2(-2)+6(5)}$ or e.g. $3(-2)^2 + 2(-2)\frac{dy}{dx} + 2\times5 + 6\times5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = ... \left(-\frac{11}{13}\right)$	M1	1.1b
	$y - 5 = "\frac{13}{11}"(x+2)$	dM1	1.1b
	$13x - 11y + 81 = 0$	A1	2.2a
		(3)	
			(7 marks)

Notes

(a) Allow equivalent notation for the $\frac{dy}{dx}$ e.g. y'

M1: Attempts to differentiate $x^3 \to ...x^2$ **and** $3y^2 \to ...y\frac{dy}{dx}$ where … are constants

B1: Correct application of the product rule on $2xy$: $2xy \to 2x\frac{dy}{dx} + 2y$

Note that some candidates have a spurious $\frac{dy}{dx} = ...$ at the start (as their intention to differentiate) and this can be ignored for the first 2 marks

M1: For a valid attempt to make $\frac{dy}{dx}$ the subject, with exactly 2 different terms in $\frac{dy}{dx}$ coming from $3y^2$ and $2xy$. Look for $(... \pm ...)\frac{dy}{dx} = ... \Rightarrow \frac{dy}{dx} = ...$ which may be implied by their working.

Condone slips provided the intention is clear.

For those candidates who had a spurious $\frac{dy}{dx} = ...$ at the start, they may incorporate this in their rearrangement in which case they will have 3 terms in $\frac{dy}{dx}$ and so score M0.

If they ignore it, then this mark is available for the condition as described above.

A1: $\frac{dy}{dx} = -\frac{2y+3x^2}{2x+6y}$ oe e.g. $\frac{dy}{dx} = \frac{-2y-3x^2}{2x+6y}$, $\frac{2y+3x^2}{-2x-6y}$ Isw once a correct expression is seen.

Note that it is sometimes unclear if the minus sign(s) is/are correctly placed and you may have to use your judgement. Evidence may be available in part (b) to help you decide if they have the correct expression.

(b)

M1: Substitutes $x = -2$ and $y = 5$ into $\frac{dy}{dx} = "-\frac{2y+3x^2}{2x+6y}"$

They must have x's and y's in their $\frac{dy}{dx}$ but condone slips in substitution provided the intention is clear.

As a minimum look for at least one x and at least one y substituted correctly.

Note that this mark may be implied by their value for $\frac{dy}{dx}$ and may be implied if, for example, they find the negative reciprocal or the reciprocal of $"-\frac{2y+3x^2}{2x+6y}"$ and then substitute x = -2 and y = 5

Alternatively, substitutes x = -2 and y = 5 into their attempt to differentiate and then rearranges to find a value or numerical expression for $\frac{dy}{dx}$

dM1: Attempts to find the equation of the normal using their gradient of the tangent and x = -2 and y = 5 correctly placed. Score for an expression of the form $(y-5)="\frac{13}{11}"(x+2)$ or if they use $y=mx+c$ they must proceed as far as c =… Must be using the **negative reciprocal** of the tangent gradient. Note that $y-5=\frac{2x+6y}{2y+3x^2}(x+2)$ is not a correct method unless the gradient is evaluated first *before* expanding.

A1: $13x-11y+81=0$ or any integer multiple of this equation including the "= 0", not just a, b, c given. e.g., $26x-22y+162=0$ is likely if they don't cancel down their gradient.

Question 9

Question	Scheme	Marks	AOs
8(a)	$R=\sqrt{2^2+8^2}=\sqrt{68}=2\sqrt{17}$	B1	1.1b
	$2\cos\theta+8\sin\theta=R\cos\theta\cos\alpha+R\sin\theta\sin\alpha$ $2=R\cos\alpha \quad 8=R\sin\alpha$ $\tan\alpha=\frac{8}{2}\Rightarrow\alpha=...$	M1	1.1b
	$\alpha=$ awrt 1.326	A1	2.2a
		(3)	
(b)(i)	$4.5\times"2\sqrt{17}"$	M1	1.1b
	$9\sqrt{17}$	A1	2.2a
(ii)	awrt 1.33	B1ft	2.2a
		(3)	
			(6 marks)

Notes

(a)

B1: $R=2\sqrt{17}$ or $\sqrt{68}$.

$\pm2\sqrt{17}$ or $\pm\sqrt{68}$ score B0

(Condone if this comes from e.g., $8=R\cos\alpha \quad 2=R\sin\alpha$)

Decimal answers score B0 unless the exact value is seen then apply isw.

M1: Proceeds to a value for α from $\tan\alpha=\pm\frac{8}{2}$, $\cos\alpha=\pm\frac{2}{"\sqrt{68}"}$, $\sin\alpha=\pm\frac{8}{"\sqrt{68}"}$

May be implied by awrt 1.33 radians or 76 degrees

A1: awrt 1.326 for α. Apply isw if this is then subsequently rounded to e.g. 1.33

(b)(i)

M1: For a value of $\pm4.5\times$ their R or allow $\pm4.5R$ (with the letter R)

But not embedded in an expression e.g. $9\sqrt{17}\cos(\theta-\alpha)$ unless extracted later.

Note that the sum may be found as $9\cos x+36\sin x$ with the maximum then found using calculus

e.g. $S=9\cos x+36\sin x\Rightarrow\frac{dS}{dx}=-9\sin x+36\cos x=0\Rightarrow\tan x=4\Rightarrow\sin x=\frac{4}{\sqrt{17}},\ \cos x=\frac{1}{\sqrt{17}}$

$\Rightarrow 9\cos x+36\sin x=9\sqrt{17}$. This will score M1 once they reach $\pm4.5\times$ their R

May be implied by $9\sqrt{17}$ or awrt 37.1 (which may come from a graphical method)

May also see e.g. $\text{Max}(9\cos x+36\sin x)=\sqrt{9^2+36^2}=...$

A1: $9\sqrt{17}$ or exact equivalent e.g. $\sqrt{1377}$, $4.5\sqrt{68}$, $4.5(2\sqrt{17})$ and apply isw once a correct answer is seen

(ii)

B1ft: awrt 1.33 (or follow through on their α even if in degrees (76), no matter how accurate)

Question 10

Question	Scheme	Marks	AOs
11(a)	$\frac{dV}{dh}=200$ oe e.g. $\frac{dh}{dV}=\frac{1}{200}$	B1	1.1b
	$\left(\frac{dh}{dt}=\right)\frac{dh}{dV}\times\frac{dV}{dt}=\frac{1}{200}\times\frac{k}{\sqrt{h}}$	M1	3.1a
	$\frac{dh}{dt}=\frac{\lambda}{\sqrt{h}}$ *	A1*	2.1
		(3)	
(b)	$\frac{dh}{dt}=\frac{\lambda}{\sqrt{h}}\Rightarrow\int h^{\frac{1}{2}}\,dh=\int\lambda\,dt\Rightarrow...h^{\frac{3}{2}}=\lambda t\{+c\}$	M1	1.1b
	$\frac{2}{3}h^{\frac{3}{2}}=\lambda t\{+c\}$ oe e.g. $\frac{h^{\frac{3}{2}}}{\frac{3}{2}}=\lambda t\{+c\}$	A1	1.1b
	$\frac{2}{3}(1.44)^{\frac{3}{2}}=\lambda\times 0+c\Rightarrow c=1.152\left(=\frac{144}{125}\right)$	dM1	3.4
	$\frac{2}{3}(3.24)^{\frac{3}{2}}=\lambda\times 8+\text{"}1.152\text{"}\Rightarrow\lambda=0.342\left(=\frac{171}{500}\right)$	ddM1	3.1b
	$h^{\frac{3}{2}}=0.513t+1.728$ oe e.g. $h^{\frac{3}{2}}=\frac{513}{1000}t+\frac{216}{125}$	A1	3.3
		(5)	
	(b) Alternative:		
	$\frac{dh}{dt}=\frac{\lambda}{\sqrt{h}}\Rightarrow\frac{dt}{dh}=\frac{\sqrt{h}}{\lambda}\Rightarrow t=...h^{\frac{3}{2}}(+c)$	M1	1.1b
	$t=\frac{2h^{\frac{3}{2}}}{3\lambda}(+c)$ oe	A1	1.1b
	$0=\frac{2(1.44)^{\frac{3}{2}}}{3\lambda}+c$ **and** $8=\frac{2(3.24)^{\frac{3}{2}}}{3\lambda}+c$ $\Rightarrow\lambda=...\left(\frac{171}{500}\right)$ **or** $c=...\left(-\frac{64}{19}\right)$	dM1	3.4
	$\Rightarrow\lambda=...\left(\frac{171}{500}\right)$ **and** $c=...\left(-\frac{64}{19}\right)$	ddM1	3.1b
	$h^{\frac{3}{2}}=0.513t+1.728$ oe e.g. $h^{\frac{3}{2}}=\frac{513}{1000}t+\frac{216}{125}$	A1	3.3
		(5)	
(c)	$5^{\frac{3}{2}}=0.513t+1.728\Rightarrow t=...$	M1	3.4
	$(t=)$ awrt18.4 min	A1	3.2a
		(2)	
			(10 marks)

Notes

(a)

B1: For $\frac{dV}{dh} = 200$ stated or used – may be implied by their chain rule attempt

M1: Requires:

- $\frac{dV}{dh} = p, \quad p > 1$
- $\frac{dV}{dt} = \pm\frac{k}{\sqrt{h}}$ or e.g. $\frac{dV}{dt} = \pm\frac{1}{k\sqrt{h}}$ (or a suitable letter for *k*, which may be λ, but must **not** be a number)
- **application** of the correct chain rule $\left(\frac{dh}{dt} =\right)\frac{dh}{dV}\times\frac{dV}{dt}$ or any equivalent with $\frac{dV}{dt} = \pm\frac{k}{\sqrt{h}}$ or $\pm\frac{1}{k\sqrt{h}}$ and their $\frac{dV}{dh}$ correctly placed. So $\frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{200k}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}}$ scores M0 as $\frac{dh}{dV}$ is incorrectly placed.

A1*: A rigorous argument with all steps shown and simplifies to achieve $\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$ with no errors.

Do not allow the use of λ for both constants. Allow use of e.g. $\frac{dV}{dt} = \pm\frac{1}{k\sqrt{h}}$ for full marks.

e.g. $\frac{dV}{dh} = 200, \quad \frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{1}{200}\times\frac{\lambda}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}}$ scores B1M1A0* *unless* e.g. "let $\lambda = \frac{\lambda}{200}$" seen.

Allow correct work leading to e.g. $\frac{dh}{dt} = \frac{1}{200}\times\frac{k}{\sqrt{h}} \to \frac{\lambda}{\sqrt{h}}$ or $\frac{dh}{dt} = \frac{1}{200}\times\frac{k}{\sqrt{h}}$ so $\lambda = \frac{k}{200}$

There must be an attempt to link the $\frac{dh}{dt}$ with the $\frac{\lambda}{\sqrt{h}}$ which may be missing an = sign.

Allow an argument with $\frac{dV}{dt} = -\frac{k}{\sqrt{h}}$ e.g. $\frac{dV}{dh} = 200, \quad \frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{1}{200}\times -\frac{k}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}}$

Withhold this A mark if there are notational errors e.g. $\frac{dV}{dt} = 200, \quad \frac{dh}{dt} = \frac{dh}{dV}\times\frac{dV}{dt} = \frac{1}{200}\times\frac{k}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}}$

scores B1(implied)M1A0*

(b) Note that some candidates may work with e.g. $\lambda = \frac{k}{200}$ **or e.g.** $\lambda = 200k$ **which is acceptable.**

Candidates who do not have a λ e.g. assume $\frac{dh}{dt} = \frac{1}{200\sqrt{h}}$ **or e.g.** $\frac{dh}{dt} = \frac{200}{\sqrt{h}}$ **then only the first 2 method marks are available (see note below). Condone use of other variables if the intention is clear but the final answer must be in terms of *h* and *t*.**

M1: Separates the variables and integrates to obtain an equation of the form $...h^{\frac{3}{2}} = \lambda t\{+c\}$ oe

The constant of integration is not needed for this mark.

A1: $\frac{2}{3}h^{\frac{3}{2}} = \lambda t(+c)$ oe. The constant of integration is not needed for this mark.

Condone spurious notation for this intermediate mark e.g. integral signs left in after integrating.

dM1: Substitutes $t = 0$ and $h = 1.44$ and attempts to find *c*.

It is dependent on the previous method mark.

Do not be concerned with the "processing" to find "*c*" as long as they are using $t = 0$ and $h = 1.44$

May be implied by their value of *c*.

ddM1: Substitutes $t = 8$ and $h = 3.24$ and their *c* and attempts to find λ. Do not be concerned with the "processing" to find λ as long as they are using $t = 8$ and $h = 3.24$.

It is dependent on both previous method marks.

A1: Correct equation in the correct form from correct work. $h^{\frac{3}{2}} = 0.513t + 1.728$ or $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$

Must follow A1 earlier so do check if this has been obtained fortuitously. Allow 1.73 for 1.728

Note candidates who do not have a λ e.g. assume $\frac{dh}{dt} = \frac{1}{200\sqrt{h}}$ or e.g. $\frac{dh}{dt} = \frac{200}{\sqrt{h}}$ can use either $t = 0$ and $h = 1.44$ **or** $t = 8$ and $h = 3.24$ to find their constant of integration.

(b)Alternative:

M1: Finds the reciprocal of both sides and integrates to obtain an equation of the form $t = ...h^{\frac{3}{2}}(+c)$

A1: $t = \frac{2h^{\frac{3}{2}}}{3\lambda}(+c)$ oe. The constant of integration is not needed for this mark.

dM1: Substitutes $t = 0$ and $h = 1.44$ **and** substitutes $t = 8$ and $h = 3.24$ **and** attempts to find λ **or** *c*.

It is dependent on the previous method mark.

Do not be concerned with the "processing" to find λ or *c* as long as they are using $t = 0$ and $h = 1.44$ and $t = 8$ and $h = 3.24$ and reach a value for λ or *c*. May be implied by their value(s).

ddM1: Complete attempt to find λ **and** *c*. **It is dependent on both previous method marks.**

Do not be concerned with the "processing" to find λ and c as long as they are using $t=0$ and $h=1.44$ and $t=8$ and $h=3.24$.

A1: Correct equation in the correct form. $h^{\frac{3}{2}}=0.513t+1.728$ or $h^{\frac{3}{2}}=\frac{513}{1000}t+\frac{216}{125}$

Must follow A1 earlier so do check if this has been obtained fortuitously. Allow 1.73 for 1.728

Special Case:

Some candidates are using the given equation in part (b) to find the value of A and the value of B using the given conditions. May score a maximum of 00110. This should be marked as follows:

M0A0: (No attempt to integrate)
M1: Substitutes $t = 0$ and $h = 1.44$ to find a value for B
dM1: Substitutes $t = 8$ and $h = 3.24$ with their value of B to find a value for A
A0: Since they have not used the given model.
(Allow full recovery in (c) if this equation is correct)

(c)

M1: Attempts to substitute $h = 5$ into their equation which must be of the form $h^{\frac{3}{2}} = At + B$ or possibly a rearranged equation e.g. $h^{\frac{1}{2}} = \sqrt[3]{At + B}$ with values of A and B leading to a value for t.

Do not be concerned about the processing as long as they use $h = 5$ and obtain a value for t even if t is negative.

A1: Awrt 18.4 minutes **following a correct equation in (b)**.

The units are required but allow e.g. min, mins, but not just 18.4 and not m (which means metres)
Allow e.g. 18 minutes 25 seconds or 18 mins 26 secs
Note this may follow A0 in part (b) as they may have rearranged incorrectly in (b) but use a correct equation in (c) e.g. $\frac{2}{3}h^{\frac{3}{2}}=\frac{171}{500}t+\frac{144}{125}$, $h^{\frac{1}{2}}=\sqrt[3]{\frac{513}{1000}t+\frac{216}{125}}$ or may come from the special case.
Apply isw following a correct time and units, e.g., 18.4 followed by 18 mins.

Question 11

Question	Scheme	Marks	AOs
12(a)	$N_A - N_B = (3+4)-(8-6) = ...$	M1	3.4
	5000 (subscribers)	A1	3.2a
		(2)	
(b)	$(T=)3$	B1	3.4
	This was the point when company A had the lowest number of subscribers	B1	2.4
		(2)	
(c)			
	$-t+7=2t+2$ o.e. or $t+1=14-2t$ o.e.	B1	3.1a
	$-t+7=2t+2$ o.e. $\Rightarrow t=...$ or $t+1=14-2t$ o.e. $\Rightarrow t=...$	M1	3.4
	One of the two critical values $t=\frac{5}{3}$ or $t=\frac{13}{3}$	A1	1.1b
	Chooses the outside region for their **two** values of t Both of $t<"\frac{5}{3}"$, $t>"\frac{13}{3}"$	A1ft	2.2a
	$\left\{t\in\mathbb{R}:t<\frac{5}{3}\right\}\cup\left\{t\in\mathbb{R}:t>\frac{13}{3}\right\}$	A1	2.5
		(5)	
(d)	The number of subscribers will become negative (when $t>7$)	B1	3.5b
		(1)	
		(10 marks)	

Notes

(a)

M1: Uses the models to find the difference when $t=0$. Allow slips in evaluating N_A and N_B but it must be clear that $t=0$ is being used. Just 5 with no working implies M1.

A1: 5000 or 5 thousand (subscribers) (5 is A0)

(b)

B1: $(t/T=)3$ Just look for the number 3 so e.g. $t>3$ or e.g. "just after 3" is acceptable.

If more than one value is offered then score B0 unless it is clear that the 3 is intended.
Must be seen in (b) not just on their diagram.

B1: Any acceptable reason e.g.

- This was the point when company A had the lowest number of subscribers
- After this point the number of subscribers started to increase
- It is the minimum
- Condone "it is the turning point"
- The graph changes direction
- It is the vertex
- The gradient becomes positive
- N_A increased

Allow this mark even if the first B mark was not scored e.g. $T=3.5$ because the graph starts to increase scores B0B1

Do not allow contradictory statements.

Do not allow:

- The graph reflects at $t=3$ on its own without further clarification

(c)
B1: Forms one valid equation (allow an equation or any inequality sign)
M1: Attempts to solve one valid equation (allow an equation or any inequality sign)
A1: For either $t=\frac{5}{3}$ or $t=\frac{13}{3}$ only (allow an equation or any inequality sign) or exact equivalent

Must be seen or used in part (c).

See notes below for attempts that use "squaring" to find the values of *t*.

A1ft: Chooses the outside region for their **two** values of t where $t > 0$.

So for $t = a$ and $t = b$ where $0 < a < b$ should be $t < a$, $t > b$. Allow , /or/and/ $\cup$ / $\cap$

Condone if incorrectly combined e.g. $"\frac{13}{3}" < t < "\frac{5}{3}"$ but **not** $"\frac{5}{3}" < t < "\frac{13}{3}"$

A1: Fully correct solution in the form $\left\{t : t < \frac{5}{3}\right\} \cup \left\{t : t > \frac{13}{3}\right\}$ or $\left\{t \middle| t < \frac{5}{3}\right\} \cup \left\{t \middle| t > \frac{13}{3}\right\}$ or

$\left(0,\ \frac{5}{3}\right) \cup \left(\frac{13}{3},\ 5\right)$ either way around but condone $\left\{t < \frac{5}{3}\right\} \cup \left\{t > \frac{13}{3}\right\}$, $\left\{t : t < \frac{5}{3} \cup t > \frac{13}{3}\right\}$,

$\left\{t < \frac{5}{3} \cup t > \frac{13}{3}\right\}$ or $\left(-\infty,\ \frac{5}{3}\right) \cup \left(\frac{13}{3},\ \infty\right)$.

It is not necessary to mention $\mathbb{R}$, e.g. $\left\{t : t \in \mathbb{R}, t > \frac{13}{3}\right\} \cup \left\{t : t \in \mathbb{R}, t < \frac{5}{3}\right\}$

Look for $\{\ \}$ and $\cup$ or condone $\left(-\infty,\ \frac{5}{3}\right) \cup \left(\frac{13}{3},\ \infty\right)$

Do not allow solutions not in set notation such as $t < \frac{5}{3}$ or $t > \frac{13}{3}$.

Note that a lower bound for $t < \frac{5}{3}$ and an upper bound for $t > \frac{13}{3}$ are not required but may be included e.g. $\left\{t \in \mathbb{R} : 0 < t < \frac{5}{3}\right\} \cup \left\{t \in \mathbb{R} : \frac{13}{3} < t < 5\right\}$ or $\left\{t \in \mathbb{R} : 0 \leqslant t < \frac{5}{3}\right\} \cup \left\{t \in \mathbb{R} : \frac{13}{3} < t \leqslant 5\right\}$

Note that the marks in this part require **valid** equations to be solved. They must have removed the mod brackets and arrived at an equation equivalent to $-t+7=2t+2$ or $t+1=14-2t$ (all you need to check initially is whether their equation **without mod brackets** is equivalent to one of these).

Note that $\left\{t : t < \frac{5}{3},\ t > \frac{13}{3}\right\}$ is condoned for the A1ft but not for the final A1.

If x is used in their set notation then final A0, but we would condone this for the penultimate A1ft.

See notes below for answers given with no working.

(d)
B1: Requires any indication that the number of subscribers will become negative. E.g.

- It allows negative subscribers (which isn't possible)
- $8-|2t-6| \geqslant 0 \Rightarrow t \leqslant 7$ so not valid after $t=7$ but condone not valid for t after (any value above 7)

But not

- Subscribers will become zero

<u>**Guidance for attempts that use "squaring" to find the values of *t* in (c):**</u>

<u>**Way 1:**</u>

$\left(-t+7\right)^2=\left(2t+2\right)^2$ o.e. or $\left(t+1\right)^2=\left(14-2t\right)^2$ o.e.	**B1**	3.1a
$\left(-t+7\right)^2=\left(2t+2\right)^2 \Rightarrow t=...$ o.e. (Gives –9 and $\frac{5}{3}$) or $\left(t+1\right)^2=\left(14-2t\right)^2 \Rightarrow t=...$ o.e. (Gives 15 and $\frac{13}{3}$)	**M1**	3.4
One of the two critical values $t=\frac{5}{3}$ or $t=\frac{13}{3}$	**A1**	1.1b
Chooses the outside region for their **two** values of t Both of $t<"\frac{5}{3}"$, $t>"\frac{13}{3}"$	**A1ft**	2.2a
$\left\{t\in\mathbb{R}:t<\frac{5}{3}\right\}\cup\left\{t\in\mathbb{R}:t>\frac{13}{3}\right\}$	**A1**	2.5

Way 2:

$\lvert t-3\rvert+4=8-\lvert 2t-6\rvert \Rightarrow \lvert t-3\rvert+\lvert 2t-6\rvert=4 \Rightarrow 3t-9=4$ o.e.	**B1**	3.1a
$\left(3t-9\right)^2=4^2 \Rightarrow 9t^2-54t+81=16 \Rightarrow 9t^2-54t+65=0 \Rightarrow t=...$(Gives $\frac{5}{3}$ and $\frac{13}{3}$)	**M1**	3.4
One of the two critical values $t=\frac{5}{3}$ or $t=\frac{13}{3}$	**A1**	1.1b
Chooses the outside region for their **two** values of t Both of $t<"\frac{5}{3}"$, $t>"\frac{13}{3}"$	**A1ft**	2.2a
$\left\{t\in\mathbb{R}:t<\frac{5}{3}\right\}\cup\left\{t\in\mathbb{R}:t>\frac{13}{3}\right\}$	**A1**	2.5

B1: Forms one valid equation and squares both sides (allow an equation or any inequality sign)
May be implied by e.g. $\left(t-3+4\right)^2=\left(8-\left(2t-6\right)\right)^2$
Alternatively, arrives at $3t-9=4$ (o.e.) as in way 2.
M1: Attempts to solve one valid equation after squaring both sides (allow an equation or any inequality sign). Note that it is acceptable to just solve $3t-9=4$
A1: As in main scheme. **A1ft:** As in main scheme. **A1:** As in main scheme.

Note: the following is common and scores 00000.

$$\lvert t-3\rvert+4=8-\lvert 2t-6\rvert \Rightarrow \left(t-3\right)^2+4=8-\left(2t-6\right)^2$$

Which typically leads to

$$t=\frac{15\pm4\sqrt{15}}{5}$$

Guidance for answers only in part (c):

$t...\text{awrt}\,1.7$ or $t...\text{awrt}\,4.3$ where … is any inequality or equation scores **11000**

$t...\frac{5}{3}$ or $t...\frac{13}{3}$ where … is any inequality or equation scores **11100** for one correct c.v.

Both $t<\text{awrt}\,1.7$ and $t>b$ where $\left\{b>\frac{5}{3}\right\}$ scores **11010** for outside region.

Both $t<a$ and $t>\text{awrt}\,4.3$ where $\left\{a<\frac{13}{3}\right\}$ scores **11010** for outside region.

Both $t<\frac{5}{3}$ and $t>b$ where $\left\{b>\frac{5}{3}\right\}$ scores **11110** for outside region with one correct.

Both $t < a$ and $t > \dfrac{13}{3}$ where $\left\{a < \dfrac{13}{3}\right\}$ scores **11110** for outside region with one correct.

Both $t < \dfrac{5}{3}$ and $t > \dfrac{13}{3}$ scores **11110** for outside region with one correct.

Fully correct e.g. $\left\{t : t < \dfrac{5}{3}\right\} \cup \left\{t : t > \dfrac{13}{3}\right\}$ scores **11111**

Question 12

Question	Scheme	Marks	AOs
13(a)	$3^{-2}\left(1+\frac{x}{3}\right)^{-2}=3^{-2}(1+...x+...x^2)$	M1	1.1b
	$(-2)\left(\frac{x}{3}\right)$ or $\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	M1	1.1b
	$\left(1+\frac{x}{3}\right)^{-2}=1+(-2)\left(\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	A1	1.1b
	$3^{-2}\left(1+\frac{x}{3}\right)^{-2}=\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$	A1	2.1
		(4)	

(a)

M1: Attempts a binomial expansion by taking out a factor of 3^{-2} or $\frac{1}{3^2}$ or $\frac{1}{9}$ and achieves at least the first 3 terms in their expansion. May be seen separately e.g. $\frac{1}{9}$ and $(1+...x+...x^2)$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)(kx)$ or $\frac{(-2)(-2-1)}{2!}(kx)^2$ where $k\neq 1$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $\left(1+\frac{x}{3}\right)^{-2}$ e.g. $=1+(-2)\left(\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2-...$ or $1-\frac{2x}{3}+\frac{x^2}{3}-...$ Do not condone missing brackets unless they are implied by subsequent work.

Condone $\left(-\frac{x}{3}\right)^2$ for $\left(\frac{x}{3}\right)^2$

Also allow this mark for 2 correct simplified terms from $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

<u>Direct expansion, if seen, should be marked as follows:</u>

$$\left((3+x)^{-2}=3^{-2}-2\times3^{-3}\times x+\frac{-2(-2-1)}{2!}\times3^{-4}\times x^2\right)$$

M1: For $(3+x)^{-2}=3^{-2}+3^{-3}\times\alpha x+3^{-4}\times\beta x^2$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)\times3^{-3}x$ or $\frac{(-2)(-2-1)}{2!}\times3^{-4}x^2$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $(3+x)^{-2}$ e.g. $3^{-2}-2\times3^{-3}\times x+\frac{-2(-2-1)}{2!}\times3^{-4}\times x^2$

Also award for at least 2 correct simplified terms from $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

Note that M0M1A1A0 is a possible mark trait in either method

<u>Note regarding a possible misread in parts (b) and (c)</u>

Some candidates are misreading $\int\frac{6x}{(3+x)^2}\,dx$ in parts (b) and (c) as $\int\frac{6}{(3+x)^2}\,dx$

<u>If parts (b) and (c) are consistently attempted</u> with $\int\frac{6}{(3+x)^2}\,dx$ then we will allow the M

marks in (b) **only**. **M1** for $x^n \to x^{n+1}$ applied to their expansion in part (a) or $6\times$ (their expansion in part (a)) and **dM1** for substituting in 0.4 and 0.2 and subtracting either way round (may be implied).
No marks are available in part (c)

MARK PARTS (b) and (c) TOGETHER

(b)	$\int 6x\times"\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)"\,\mathrm{d}x = \int\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)\mathrm{d}x = ...$	**M1**	1.1b
	$\int\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)\mathrm{d}x = \frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18}$ oe	**A1**	1.1b
	$\left["\frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18}"\right]_{0.2}^{0.4} = \left(\frac{(0.4)^2}{3}-\frac{4(0.4)^3}{27}+\frac{(0.4)^4}{18}\right)-\left(\frac{(0.2)^2}{3}-\frac{4(0.2)^3}{27}+\frac{(0.2)^4}{18}\right)$	**dM1**	3.1a
	$=$ awrt 0.03304 or $\frac{223}{6750}$	**A1**	1.1b
		(4)	

(b)
M1: Attempts to multiply their expansion from part (a) by $6x$ or just x and attempts to integrate. Condone copying slips and slips in expanding. Look for $x^n \to x^{n+1}$ at least once having multiplied by $6x$ or x. Ignore e.g. spurious integral signs.
A1: Correct integration, simplified or unsimplified.

$$\frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18} \text{ oe e.g. } \frac{1}{9}\left(3x^2-\frac{4x^3}{3}+\frac{x^4}{2}\right),\ 6\left(\frac{x^2}{18}-\frac{2x^3}{81}+\frac{x^4}{108}\right)$$

If they have extra terms they can be ignored.
Ignore e.g. spurious integral signs.
dM1: An overall problem-solving mark for
- using part (a) by integrating $6x\times$ their binomial expansion and
- substituting in 0.4 and 0.2 and subtracting either way round (may be implied)

For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the limits 0.4 and 0.2.
This could be e.g. $\left[\mathrm{f}(x)\right]_{0.2}^{0.4} = ...$ provided the first M was scored. If the integration was correct, evidence can be taken from answer of awrt 0.0330 if limits are not seen elsewhere. **Depends on the first M mark.**
A1: awrt 0.03304 (NB allow the exact value which is $\frac{223}{6750} = 0.033037037...$).
Isw following a correct answer.
Note answers which use additional terms in the expansion to give a different approximation score A0
Also note that the actual value is 0.032865…

Some may use integration by parts in (b) and the following scheme should be applied.
Integration by parts in (b):

Either by taking $u = 6x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = "\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)"$

$$\int 6x\times"\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)"\mathrm{d}x = 6x\times\left(\frac{1}{9}x-\frac{x^2}{27}+\frac{x^3}{81}\right)-6\int\left(\frac{1}{9}x-\frac{x^2}{27}+\frac{x^3}{81}\right)\mathrm{d}x$$

$$= 6x\left(\frac{1}{9}x-\frac{x^2}{27}+\frac{x^3}{81}\right)-\left(\frac{1}{3}x^2-\frac{2x^3}{27}+\frac{6x^4}{324}\right)$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x\times"\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)"\mathrm{d}x = kx\times\mathrm{f}(x)-k\int\mathrm{f}(x)\mathrm{d}x = kx\times\mathrm{f}(x)-k\mathrm{g}(x)$$

Where f(x) is an attempt to integrate their expansion from (a) with $x^n \to x^{n+1}$ at least once **and** g(x) is an attempt to integrate their f(x) with $x^n \to x^{n+1}$ at least once
A1: Fully correct integration. Then **dM1A1** as in the main scheme

Or by taking $u = "\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)"$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 6x$

$$\int 6x \times "\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)" \mathrm{d}x = 3x^2 \times \left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right) - \int 3x^2 \times \left(-\frac{2}{27}+\frac{2x}{27}\right)\mathrm{d}x$$

$$= 3x^2 \times \left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right) - \int\left(\frac{6x^3}{27}-\frac{6x^2}{27}\right)\mathrm{d}x = 3x^2 \times \left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right) - \left(\frac{6x^4}{108}-\frac{6x^3}{81}\right)$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times "\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)" \mathrm{d}x = kx^2 \times \mathrm{f}(x) - k\int x^2 \mathrm{g}(x)\mathrm{d}x = kx^2 \times \mathrm{f}(x) - k\mathrm{h}(x)$$

Where f(x) is their expansion from (a) and g(x) is an attempt to differentiate their f(x) with $x^n \to x^{n-1}$ at least once **and** h(x) is an attempt to integrate their $x^2\mathrm{g}(x)$ with $x^n \to x^{n+1}$ at least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme

(c)	Overall problem-solving mark (see notes)	**M1**	3.1a
	$u = 3+x \Rightarrow \int_{3.2}^{3.4} \mathrm{f}(u)\,\mathrm{d}u \Rightarrow \int_{3.2}^{3.4} \frac{6(u-3)}{u^2}\,\mathrm{d}u = \int_{3.2}^{3.4} \frac{6}{u}-\frac{18}{u^2}\mathrm{d}u \Rightarrow ...\ln u + ...u^{-1}$	**M1**	1.1b
	$\int_{3.2}^{3.4} \frac{6(u-3)}{u^2}\,\mathrm{d}u = \int_{3.2}^{3.4} \frac{6}{u}-\frac{18}{u^2}\mathrm{d}u \Rightarrow 6\ln u + 18u^{-1}$	**A1**	1.1b
	$\left[6\ln u + 18u^{-1}\right]_{3.2}^{3.4} = \left(6\ln 3.4 + \frac{18}{3.4}\right) - \left(6\ln 3.2 + \frac{18}{3.2}\right) = ...$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1
		(5)	
(c) Alt 1	Overall problem-solving mark (see notes)	**M1**	3.1a
	$\int 6x(3+x)^{-2}\,\mathrm{d}x = \frac{...x}{3+x} \pm ...\int (3+x)^{-1}\mathrm{d}x = \frac{...x}{3+x} \pm ...\ln(3+x)$ oe	**M1**	1.1b
	$= 6\ln(3+x) - \frac{6x}{3+x}$ oe	**A1**	1.1b
	$\left(6\ln(3+0.4) - \frac{6(0.4)}{3+0.4}\right) - \left(6\ln(3+0.2) - \frac{6(0.2)}{3+0.2}\right) = ...$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1

(c) Alt 2	Overall problem-solving mark (see notes)	**M1**	3.1a
	$\int 6x(3+x)^{-2}\,\mathrm{d}x = \int\left(\frac{...}{(3+x)} + \frac{...}{(3+x)^2}\right)\mathrm{d}x = ...\ln(3+x) + \frac{...}{3+x}$ oe	**M1**	1.1b
	$= 6\ln(3+x) + \frac{18}{3+x}$ oe	**A1**	1.1b
	$\left(6\ln(3+0.4) + \frac{18}{3+0.4}\right) - \left(6\ln(3+0.2) + \frac{18}{3+0.2}\right) = ...$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1
			(13 marks)

Notes

(c) There are various methods which can be used

M1: An overall problem-solving mark for **all of**

- using an appropriate integration technique e.g. substitution, by parts or partial fractions – note that this may not be correct but mark positively if they have tried one of these approaches
- integrates one of their terms to a natural logarithm, e.g., $\frac{a}{3+x} \to b\ln(3+x)$ or $\frac{a}{u} \to b\ln u$
- substitutes in correct limits and subtracts either way round

M1: Integrates to achieve an expression of the required form for their chosen method

- substitution: $u = x+3 \to \pm\frac{a}{u} \pm b\ln u$ or e.g. $u = (x+3)^2 \to \pm\frac{a}{\sqrt{u}} \pm b\ln u$

- parts: $\pm a\ln(3+x)\pm\dfrac{bx}{3+x}$ condone missing brackets e.g. $...\ln x+3$ for $...\ln(3+x)$
- partial fractions: $\pm a\ln(3+x)\pm\dfrac{b}{3+x}$ condone missing brackets e.g. $...\ln 3+x$ for $...\ln(3+x)$

A1: Correct integration for their method e.g.

- substitution: $u=x+3\to 6\ln u+18u^{-1}$ or e.g. $u=(x+3)^2\to 3\ln u+\dfrac{18}{\sqrt{u}}$
- parts: $6\ln(3+x)-\dfrac{6x}{3+x}$
- partial fractions: $6\ln(3+x)+\dfrac{18}{3+x}$ oe e.g. $3\ln(9+6x+x^2)+\dfrac{18}{3+x}$

Note that the above terms may appear "separated" but must be correct with the correct signs.
(ignore any reference to a constant of integration)
Do not condone missing brackets e.g. $6\ln x+3$ for $6\ln(3+x)$ unless they are implied by later work.

ddM1: Substitutes in the correct limits for their integral and subtracts either way round to find a value
Depends on both previous method marks.
For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the appropriate limits.
This could be e.g. $\left[f(x)\right]_{0.2}^{0.4}=...$ provided both previous M marks were scored.
Note that for substitution they may revert back to $3+x$ and so should be using 0.4 and 0.2

A1: A full and rigorous argument leading to $6\ln\left(\dfrac{17}{16}\right)-\dfrac{45}{136}$ or exact equivalent e.g. $3\ln\left(\dfrac{289}{256}\right)-\dfrac{45}{136}$ or e.g. $-6\ln\left(\dfrac{16}{17}\right)-\dfrac{45}{136}$

The brackets are not required around the $\dfrac{17}{16}$ and allow exact equivalents e.g. allow 1.0625 or $1\dfrac{1}{16}$ but not e.g. $\dfrac{3.4}{3.2}$. The $\dfrac{45}{136}$ must be exact or an exact equivalent. Also allow e.g. $6\ln\left|\dfrac{17}{16}\right|-\dfrac{45}{136}$
Ignore spurious integral signs that may appear as part of their solution.

Question 13

Question	Scheme	Marks	AOs
14(a)	e.g. $2\frac{\sin\theta}{\cos\theta}(8\cos\theta+23(1-\cos^2\theta))=8\times 2\sin\theta\cos\theta\sec^2\theta$	B1	1.2
	$2\tan\theta(8\cos\theta+23\sin^2\theta)=8\sin 2\theta\sec^2\theta$ $\Rightarrow 2\sin\theta\cos\theta(8\cos\theta+23(1-\cos^2\theta))=8\sin 2\theta$ $\sin 2\theta(8\cos\theta+23(1-\cos^2\theta))=8\sin 2\theta$ $\sin 2\theta(23\cos^2\theta-8\cos\theta-15)=0$	M1A1	2.1 2.2a
		(3)	
(b)	$\sin 2x(23\cos^2 x-8\cos x-15)=0$		
	$\sin 2x=0\Rightarrow x=360°$ **or** $540°$	B1	2.2a
	$23\cos^2 x-8\cos x-15\Rightarrow \cos x=-\frac{15}{23}$	M1	1.1b
	$\cos x=-\frac{15}{23}\Rightarrow x=...$	dM1	1.1b
	$x=360°, 540°$ and awrt $491°$ only	A1	2.3
		(4)	
			(7 marks)

Notes

(a) Allow use of e.g. x but the final mark requires the equation to be in terms of θ

B1(M1 on EPEN): For recalling and using at least one correct trigonometric identity in the given equation.

e.g. one of: $\sin^2\theta+\cos^2\theta=1$, $1+\tan^2\theta=\sec^2\theta$, $\tan\theta=\frac{\sin\theta}{\cos\theta}$, $\sin 2\theta=2\sin\theta\cos\theta$

This may be seen explicitly or may be implied by their working by e.g. $\tan\theta\cos\theta=\sin\theta$ or they might multiply both sides by $\cos^2\theta$ leaving $8\sin 2\theta$ on the rhs implying $1+\tan^2\theta=\sec^2\theta$

M1: For manipulating the equation using trigonometric identities (condoning sign slips only in the identities and arithmetic slips) to obtain an expression of the form:

$A\sin 2\theta\cos^2\theta+B\sin 2\theta\cos\theta+C\sin 2\theta\ (=0)$ or $\sin 2\theta\left(A\cos^2\theta+B\cos\theta+C\right)\ (=0)$ with $A, B, C\neq 0$

A1: $\sin 2\theta(23\cos^2\theta-8\cos\theta-15)=0$ oe e.g. $\sin 2\theta(-23\cos^2\theta+8\cos\theta+15)=0$ cao

Note that this is not a given answer so condone notational slips e.g. $\cos\theta^2$ for $\cos^2\theta$ provided the intention is clear but the final equation must have no notational errors.

Note that the "= 0" is not required for the M1 but is required for the A1

Note: some candidates arrive at the correct final answer fortuitously following errors in their work.

(b) Allow all marks in (b) to score if the correct equation is obtained fortuitously in part (a)

Also allow use of θ instead of x throughout in part (b). Correct answers, no working scores max 1000

B1: For one of $x=360(°)$ or $x=540(°)$ Condone $x=2\pi$ or $x=3\pi$ for this mark.

The degrees symbol is not required. This may come from $\cos x=1$

M1: Attempts to solve their 3TQ from part (a) or a "made up" 3TQ (which may only be seen in (b)) leading to a value for $\cos x$. The general guidance for solving a 3 term quadratic equation can be applied. Allow solution(s) from a calculator which may be implied by at least one correct value for their 3TQ. Must be a value for $\cos x$ and not e.g. x.

dM1: Attempts to find one of their angles in the range $360<x<540$ (but not 450) for their $\cos x=k$ where $|k|<1$ May be implied by their value(s) but must be in degrees.

Requires them to state a value for $\cos x$. Must be checked (you can check cos(their x) = their k (1sf))

A1: $x=360°, 540°$ and awrt $491°$ only with no other values in range (including 450).

The degrees symbol is not required. awrt 491 must come from $\cos x=-\frac{15}{23}$